The 1-Discontinuous Lambek Calculus:

Type Logical Grammar and discontinuity in natural language

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Introduction

This work presents a natural extension of the (associative) Lambek Calculus (LC) [Lambek 1958]. The type-logical approach to language is basically categorial grammar formulated on the foundations of type theory and substructural logics. So, the LC is a logic with its proof-theory and (prosodic) interpretation, which enjoys several interesting (and beautiful) mathematical properties such as cut-elimination and interpolation. A fundamental feature of the LC is the so-called Curry-Howard homomorphism, which gives to every proof-derivation a lambda-term encoding its semantics. This is what linguists call the syntax-semantics interface. However, the LC, despite covering several interesting linguistic phenomena, such as non-constituent coordination, encounters some difficulties with phenomena involving discontinuity. As a matter fact, the LC is claimed to be the logic of (only) concatenation.

Our main contribution is the focus on what we call the 1-Discontinuous Lambek Calculus (1-**DLC**, which is a fragment of calculi considered in [Morrill and Merenciano96] and [Morrill2002]), an extension of the LC (three new binary connectives and two units[1]) which is able to deal with several discontinuous linguistic phenomena, such as particle verbs, medial extraction, cross-serial dependencies, gapping and quantifier raising. Pentus [Pentus 1993] proved that the LC is weakly equivalent to context-free grammars (the so-called Chomsky conjecture). We give some results that show that (a fragment) of 1-**DLC** is weakly equivalent to Head Grammars [Pollard 84]. Head Grammars are a proper extension of context-free grammars. They are a proper subclass of mildly context-sensitive grammars, which are believed by several linguists to be the right formal system to describe natural language phenom-

[1] Two unary connectives, bridge and split are considered as well.

ena, including of course, discontinuity. Moreover, the LC is complete w.r.t the class of free monoids (proof by [Pentus 1995]). Here again, we give a partial completeness result (completeness for the continuous and discontinuous implicative fragment) w.r.t. the class of free 1-graded monoids, which are an extension of monoids. We prove also the full completeness of the 1-**DLC** w.r.t the class of preordered 1-graded monoids.

Moreover, the 1-**DLC** is presented model-theoretically with a categorical calculus of sorted types. Then, a new kind of sequent calculus, the hypersequent calculus (see [Morrill98]) is given to translate the categorical calculus to a logic with elimination rules and introduction rules and without structural rules (except an implicit associativity). The hypersequent calculus intuitively shows that types can be inhabited by pairs of strings (as in Head Grammars). The fundamental cut elimination theorem for the hypersequent calculus is proved.

The 1-**DLC** and the hypersequent calculus were invented by Glyn Morrill [Morrill 1998]. The type-logical study of discontinuity was initiated by Moortgat [Moortgat 1988]. Morrill aspires to give the best type-logical account of discontinuity, by means of a *separator* or point of discontinuity which is in some sense close to a proposal by Versmissen [Ver91] and Solias [Solias 1992]. Morrill crucially interprets types with sorted operations (linear functions of strings: concatenation and wrapping). This is an important feature of our model-theoretical results.

The first chapter introduces the 1-discontinuous *categorical* Lambek calculus of sorted types, and its translation to the hypersequent calculus or 1-**DLC** with or without units. The second chapter is devoted to the proof-theory of the new calculus, whereas the next chapter considers prosodic interpretation. Finally, the last chapter presents linguistic applications to discontinuity and studies the weak generative capacity of the 1-**DLC**.

Chapter 1

From sequent calculus to hypersequent calculus

In this chapter, the fundamental sorted operations with strings and tuples of strings are considered. We present sorted types and the 1-discontinuous categorical Lambek calculus. The natural way to interpret sorted types is in 1-graded monoids which are studied in depth. Finally, we formulate a syntactic translation from the 1-discontinuous categorical Lambek calculus to the hypersequent calculus or 1-**DLC**ϵ.

1.1 Types, sequents, hypersequents

1.1.1 Strings and tuples of strings

Let V_0 be a set, closed by an internal binary operation $\cdot$, with a distinguished element $\mathbf{I}$, called the unit. The structure $\langle V_0, \cdot, \mathbf{I}\rangle$ is called a *monoid* if the following axioms hold:

- $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- $x \cdot I = I \cdot x = x$

Let \$ be an element not belonging to V_0. Let $\langle V_0, \$\rangle$ be the monoid freely generated by V_0 and \$. We define $V_1 \subseteq \langle V_0, \$\rangle$ as:

$$V_1 = \{a \cdot \$ \cdot b : a, b \in V_0\}$$

Remark 1. \$ *is called the separator.*

As $\$ \notin V_0$, it is easy to see that V_1 is in a natural bijection with the set of pairs of strings of V_0, $V_0 \times V_0$:

$$\begin{aligned} &V_0 \times V_0 \xrightarrow{\simeq} V_1 \\ &\overrightarrow{x} = (x_1, x_2) \mapsto x_1 \cdot \$ \cdot x_2 \end{aligned}$$

Definition 1. *An element of V_0 is said to be of sort 0. An element of V_1 is said to be of sort 1.*

Let $\widetilde{V} \stackrel{def}{=} V_0 \cup V_1$[1]. We define in $\widetilde{V}$ two sort polymorphic binary operations **conc** and **wrap**:

[1] $\widetilde{V} \subsetneq \langle V_0, \$\rangle$, for $\langle V_0, \$\rangle$ contains strings with an unbounded number of occurrences of \$.

$$
\begin{aligned}
&\mathbf{conc_{(0,0)\to 0}} : V_0 \times V_0 \overset{(0,0)\to 0}{\longrightarrow} V_0 \\
&(s,t) \mapsto s \cdot t \\
&\mathbf{conc_{(1,0)\to 1}} : V_1 \times V_0 \overset{(1,0)\to 1}{\longrightarrow} V_1 \\
&(s_1 \cdot \$ \cdot s_2, t) \mapsto s_1 \cdot \$ \cdot s_2 \cdot t \\
&\mathbf{conc_{(0,1)\to 1}} : V_0 \times V_1 \overset{(0,1)\to 1}{\longrightarrow} V_1 \\
&(t, s_1 \cdot \$ \cdot s_2) \mapsto t \cdot s_1 \cdot \$ \cdot s_2 \\
&\mathbf{wrap_{(1,0)\to 0}} : V_1 \times V_0 \overset{(1,0)\to 0}{\longrightarrow} V_0 \\
&(s_1 \cdot \$ \cdot s_2, t) \mapsto s_1 \cdot t \cdot s_2 \\
&\mathbf{wrap_{(1,1)\to 1}} : V_1 \times V_1 \overset{(1,1)\to 1}{\longrightarrow} V_1 \\
&(s_1 \cdot \$ \cdot s_2, t_1 \cdot \$ \cdot t_2) \mapsto s_1 \cdot t_1 \cdot \$ \cdot t_2 \cdot s_2
\end{aligned}
$$

$\mathbf{wrap_{(1,0)\to 0}}$ and $\mathbf{wrap_{(1,1)\to 1}}$ have disjoint domains, therefore it is correct to define the sort polymorphic function **wrap**:

$$\mathbf{wrap} \overset{def}{=} \mathbf{wrap_{(1,0)\to 0}} \uplus \mathbf{wrap_{(1,1)\to 1}}$$

$\mathbf{conc_{(0,0)\to 0}}$, $\mathbf{conc_{(1,0)\to 1}}$ and $\mathbf{conc_{(0,1)\to 1}}$ have also disjoint domains, so again it is correct to define the sort polymorphic function **conc**:

$$\mathbf{conc} \overset{def}{=} \mathbf{conc_{(0,0)\to 0}} \uplus \mathbf{conc_{(1,0)\to 1}} \uplus \mathbf{conc_{(0,1)\to 1}}$$

For the sake of simplicity and readability, **wrap** and **conc** will be denoted as infix operators $\hat{\cdot}$ and $\cdot$ respectively. Another useful convention is to denote elements of sort 1 as vectors of strings, e.g., $\vec{s} = s_1 \cdot \$ \cdot s_2$.

Definition 2. *I and J denote the unit elements for* **conc** *and* **wrap**. *I is of sort 0, and J of sort 1.*

- $(\mathbf{Unit_1})$ $s \cdot I = I \cdot s = s$
- $(\mathbf{Unit_2})$ $\vec{s} \cdot I = I \cdot \vec{s} = \vec{s}$
- $(\mathbf{Unit_3})$ $\vec{s} \,\hat{\cdot}\, J = J \,\hat{\cdot}\, \vec{s} = \vec{s}$

Now, we give some properties which hold of $\widetilde{V}$:

Lemma 1.

- $(\mathbf{Ass_{conc_1}})$ $s \cdot (t \cdot r) = (s \cdot t) \cdot r$
- $(\mathbf{Ass_{conc_2}})$ $\vec{s} \cdot (t \cdot r) = (\vec{s} \cdot t) \cdot r$
- $(\mathbf{Ass_{conc_3}})$ $s \cdot (\vec{t} \cdot r) = (s \cdot \vec{t}) \cdot r$
- $(\mathbf{Ass_{conc_3}})$ $s \cdot (t \cdot \vec{r}) = (s \cdot t) \cdot \vec{r}$
- $(\mathbf{Ass_{wrap_1}})$ $\vec{s} \mathbin{\hat{\cdot}} (\vec{t} \mathbin{\hat{\cdot}} r) = (\vec{s} \mathbin{\hat{\cdot}} \vec{t}) \mathbin{\hat{\cdot}} r$
- $(\mathbf{Ass_{wrap_2}})$ $\vec{s} \mathbin{\hat{\cdot}} (\vec{t} \mathbin{\hat{\cdot}} \vec{r}) = (\vec{s} \mathbin{\hat{\cdot}} \vec{t}) \mathbin{\hat{\cdot}} \vec{r}$
- $(\mathbf{MixedAss1})$ $s \cdot (\vec{t} \mathbin{\hat{\cdot}} r) = (s \cdot \vec{t}) \mathbin{\hat{\cdot}} r$
- $(\mathbf{MixedAss2})$ $s \cdot (\vec{t} \mathbin{\hat{\cdot}} \vec{r}) = (s \cdot \vec{t}) \mathbin{\hat{\cdot}} \vec{r}$
- $(\mathbf{SplitWrap_0})$ $a \cdot b \cdot c = (a \cdot J \cdot c) \mathbin{\hat{\cdot}} b$
- $(\mathbf{SplitWrap_1})$ $a \cdot \vec{b} \cdot c = (a \cdot J \cdot c) \mathbin{\hat{\cdot}} \vec{b}$

Proof. Trivial. □

We are now in a position to formulate a natural algebraic extension to monoids, which are the natural model-theoretical foundation of continuity. This structure gives us the model-theoretical foundation of discontinuity. Let's formulate an abstract structure which captures the properties enunciated in the last lemma.

Definition 3 (1-graded monoid)**.** *A sorted algebra* $\langle U_0 \cup U_1, \cdot, \mathbin{\hat{\cdot}}, \mathbf{I}, \mathbf{J} \rangle$ *is called a* ***1-graded monoid*** *when the properties of the last lemma and the unit properties* $\mathbf{Unit}_i$ *hold.*

Remark 2. *The class of 1-graded monoids is not empty, for* $\langle \widetilde{V}, \cdot, \mathbin{\hat{\cdot}}, I, J \rangle$ *is an example of a 1-graded monoid.*

1.2 Towards sorted types

We want now to define some natural algebraic operations on the power set $\mathcal{P}(\widetilde{V})$, of a 1-graded monoid. Let $A, \ldots, Z$ denote arbitrary subsets of $\widetilde{V}$.

Definition 4 (**principle of well-sorting**). *Given $A \subseteq \widetilde{V}$, we say that A is sorted if:*

$$A \subseteq V_0 \textit{ or } A \subseteq V_1$$

This means that sorted subsets of 1-graded monoids are inhabited by elements of the same sort.

We define six sorted operations $\circ$, $\hat{\circ}$ $, \backslash\backslash, //, \Uparrow, \Downarrow$ on sorted subsets of $\widetilde{V}$. As expected, these operations are sort polymorphic, and by the **principle of well-sorting** their sort is inferred by the sort of their arguments. The counterpart of the units I, J in $\mathcal{P}(\widetilde{V})$ are also defined:

- Continuous connectives:

$$A \circ B = \{d : \exists a \in A\ \exists b \in B \text{ such that } d = a \cdot b\}$$

- $B \circ A$ is of sort 0, if A and B are of sort 0.
- $B \circ A$ is of sort 1, if A is of sort 0 and B of sort 1.
- $B \circ A$ is of sort 1, if A is of sort 1 and B of sort 0.

$$B//A = \{d | \forall a \in A\ d \cdot a \in B\}$$

- $B//A$ is of sort 0, if A and B are of sort 0.
- $B//A$ is of sort 1, if A is of sort 0 and B of sort 1.
- $B//A$ is of sort 0, if A and B are of sort 1.

$$A\backslash\backslash B = \{d | \forall a \in A \; a \cdot d \in B\}$$

- $A\backslash\backslash B$ is of sort 0, if A and B are of sort 0.
- $A\backslash\backslash B$ is of sort 1, if A is of sort 0 and B of sort 1.
- $A\backslash\backslash B$ is of sort 0, if A and B are of sort 1.

- Discontinuous connectives:

$$A \hat{\circ} B = \{d : \exists \alpha \in V_1 \cap A, \exists b \in B \text{ such that } d = \alpha \,\hat{:}\, b\}$$

- $A \hat{\circ} B$ is of sort 0, if A is of sort 1 and B of sort 0.
- $A \hat{\circ} B$ is of sort 1, if A is of sort 1 and B of sort 1.

$$B \Uparrow A = \{\delta | \delta \in V_1, \& \forall a \in A \; \delta \,\hat{:}\, a \in B\}$$

$B \Uparrow A$ is always of sort 1.

$$A \Downarrow B = \{\delta | \forall \alpha_1 \in V_1 \cap A \; \alpha \,\hat{:}\, \delta \in B\}$$

- $A \Downarrow B$ is of sort 0, if A is of 1 and B of sort 0.
- $A \Downarrow B$ is of sort 1, if A and B are of sort 1.

$$\check{}A = \{a \cdot \$ \cdot b : a \cdot b \in A\}, \text{ if A of sort 0}$$

$$\hat{}A = \{a \cdot b : a \cdot \$ \cdot b \in A\}, \text{ if A of sort 1}$$

$$\mathbb{I} = \{I\}$$

$$\mathbb{J} = \{I \cdot \$ \cdot I\} = \{\$\}$$

Remark 3. *The unary operations* ˇ *and* ˆ *are called* split *and* bridge *respectively.*

1.2.1 Algebraic properties

Let's explore some fundamental algebraic properties defined on $\mathcal{P}(\widetilde{V})$. The two sorted linear operations **conc** and **wrap** generate sorted associative residuated triples, $(\circ, \backslash\backslash, //)$ and $(\hat{\circ}, \Uparrow, \Downarrow)$. Let's see them:

- Continuous residuation:

$$\mathbf{Res_{conc}} \begin{array}{l} A \circ B \subseteq C \textit{ iff } B \subseteq A\backslash\backslash C \\ A \circ B \subseteq C \textit{ iff } A \subseteq C//B \end{array}$$

Discontinuous residuation:

$$\mathbf{Res_{wrap}} \begin{array}{l} A \hat{\circ} B \subseteq C \textit{ iff } B \subseteq A \Downarrow C \\ A \hat{\circ} B \subseteq C \textit{ iff } A \subseteq C \Uparrow B \end{array}$$

Associative rules:

$$\mathbf{Ass_{con}}\ A \circ (B \circ C) = (A \circ B) \circ C$$

$$\mathbf{Ass_{wrap}}\ A \hat{\circ} (B \hat{\circ} C) = (A \hat{\circ} B) \hat{\circ} C$$

$$\mathbf{MixedAss}\ A \circ (B \hat{\circ} C) = (A \circ B) \hat{\circ} C$$

$$\mathbf{Units}\ A \circ \mathbb{I} = \mathbb{I} \circ A = A \quad A \hat{\circ} \mathbb{J} = \mathbb{J} \hat{\circ} A = A$$

Moreover, following Morrill [1994], a fundamental property called the split-wrap rule is considered:

SplitWrap $A \circ B \circ C = (A \circ \mathbb{J} \circ C) \hat{\circ} B$, (A, C of sort 0, and B are of sort 0 or sort 1)

These algebraic properties are to be read carefully: sorts must be appropriate in order to have well defined operations. So the reader is invited to check the sorts of the laws displayed above.

1.2.2 A categorical 1-discontinuous syntactic calculus

Following Lambek [58], we present a categorical calculus of types, which allows us to deal with discontinuous phenomena of natural language. Types are (logical) formulas which are intuitively interpreted as before, that is, they are inhabited by elements of a 1-graded monoid. This categorical calculus is mapped into the discontinuous hypersequent calculus, a pure substructural logic which is a natural extension of the so-called (associative) Lambek Calculus. The hypersequent calculus for discontinuity was invented by Morrill [Morrill 97, 2003], and we call our fragment 1-**DLC**. Morrill has been working on a number of papers (and now, in a joint work with Fadda and the author) on the generalized discontinuous Lambek Calculus, or ω-**DLC**, a pure substructural logic which admits the existence of multiple points of discontinuity. As a matter of fact, the reader has probably expected that **conc** and **wrap** can be infinitely sorted in a ω-graded monoid.

Definition 5 (Set of discontinuous Lambek types)**.** *Let $\mathcal{A}_0$ and $\mathcal{A}_1$ be sets of atomic types of sort 0 and 1 respectively. We define the sorted set $\mathcal{F}$ of sorted types as:*

$$
\begin{aligned}
\mathcal{F}_0 &::= I \,|\, \mathcal{A}_0 \,|\, \mathcal{F}_0\backslash\mathcal{F}_0 \,|\, \mathcal{F}_1\backslash\mathcal{F}_1 \,|\, \mathcal{F}_0/\mathcal{F}_0 \,|\, \mathcal{F}_1/\mathcal{F}_1 \,|\, \mathcal{F}_0 \bullet \mathcal{F}_0 \,|\, \mathcal{F}_1 \downarrow \mathcal{F}_0 \,|\, \mathcal{F}_1 \odot \mathcal{F}_0 \\
\mathcal{F}_1 &::= J \,|\, \mathcal{A}_1 \,|\, \mathcal{F}_0 \uparrow \mathcal{F}_0 \,|\, \mathcal{F}_1 \uparrow \mathcal{F}_1 \,|\, \mathcal{F}_1 \downarrow \mathcal{F}_1 \,|\, \mathcal{F}_0\backslash\mathcal{F}_1 \,|\, \mathcal{F}_1/\mathcal{F}_0 \,|\, \mathcal{F}_0 \bullet \mathcal{F}_1 \,|\, \mathcal{F}_1 \bullet \mathcal{F}_0 \,|\, \mathcal{F}_1 \odot \mathcal{F}_1 \\
\mathcal{F} &::= \mathcal{F}_0 \,|\, \mathcal{F}_1
\end{aligned}
$$

As the reader may notice, this recursive definition of sorted types mimics the six operations $\circ$, $\hat{\circ}$, $\backslash\backslash, //, \Uparrow, \Downarrow$ on the power set of 1-graded monoids, as well as the units $\mathbb{I}, \mathbb{J}$, and in the same manner, the sort of a type can be inferred also from its subtypes. We present now the categorical 1-discontinuous syntactic calculus, a natural extension of the categorical (continuous) syntactic Lambek calculus in figure 1.1. Obviously, type constructors $\backslash, /, \bullet, \uparrow, \downarrow, \odot$ are sort polymorphic.

Definition 6. *A type inference of the categorical 1-discontinuous syntactic calculus is called a sequent. A provable sequent $\mathcal{S} = A \Rightarrow B$, is a (finite) derivation in the categorical calculus whose last conclusion is $\mathcal{S}$: in that case we write $\vdash A \Rightarrow B$.*

Type inferences in the categorical 1-discontinuous syntactic calculus are sound modulo sort, i.e., the reader must infer the right sort of types in order to have meaningful sequents.

$$\frac{}{A \Rightarrow A}\ id_0 \qquad\qquad \frac{}{A \Rightarrow A}\ id_1$$

$$\textbf{Res}_{\textbf{cont}}\ A \bullet B \Rightarrow C \text{ iff } B \Rightarrow A\backslash C \qquad A \bullet B \Rightarrow \text{ iff } A \Rightarrow C/B$$

$$\textbf{Res}_{\textbf{discont}}\ A \odot B \Rightarrow C \text{ iff } B \Rightarrow A \downarrow C \qquad A \odot B \Rightarrow C \text{ iff } A \Rightarrow C \uparrow B$$

$$\textbf{Ass}_{con}\ A \bullet (B \bullet C) \Rightarrow E \text{ iff } (A \bullet B) \bullet C \Rightarrow E \qquad \textbf{Ass}_{wrap}\ A \odot (B \odot C) \Rightarrow E \text{ iff } (A \odot B) \odot C \Rightarrow E$$

$$\textbf{Unit}_0\ A \Leftrightarrow A \bullet I \Leftrightarrow I \bullet A \qquad \textbf{Unit}_1\ A \Leftrightarrow A \odot J \Leftrightarrow J \odot A$$

$$\textbf{MixAss}\ A \bullet (B \odot C) \Rightarrow E \text{ iff } (A \bullet B) \odot C \Rightarrow E$$

$$\textbf{SplitWrap}\ (A \bullet B \bullet C) \Rightarrow E \text{ iff } (A \bullet J \bullet C) \odot B \Rightarrow E \text{ with A, C of sort 0, and B of sort 0 or 1}$$

$$\textbf{Cut}\ A \Rightarrow B \text{ and } B \Rightarrow C \text{ then } A \Rightarrow C \text{ where } A, B \text{ and } C \text{ are all of the same sort}$$

Figure 1.1: Categorical 1-discontinuous syntactic calculus

Let's give some derivable type inferences in the categorical 1-discontinuous calculus:

- Continuous and discontinuous modus ponens (MP):

$$(B/A) \bullet A \Rightarrow B, \text{ for by residuation } \dfrac{\dfrac{}{B/A \Rightarrow B/A}\ Id_0}{(B/A) \bullet A \Rightarrow B}\ Res_{cont}$$

  Similar reasoning for $\backslash$.

$$(B \uparrow A) \odot A \Rightarrow B, \text{ for by discontinuous residuation } \dfrac{\dfrac{}{B \uparrow A \Rightarrow B \uparrow A}\ Id_0}{(B \uparrow A) \odot A \Rightarrow B}\ Res_{discont}$$

  Similar reasoning for $\downarrow$.

- Type lifting:

$$\dfrac{\dfrac{}{N \bullet (N\backslash S) \Rightarrow S}\ MP}{N \Rightarrow S/(N\backslash S)}\ Res_{cont}$$

$$\dfrac{\dfrac{}{(S \uparrow N) \odot N \Rightarrow S}\ MP}{N \Rightarrow (S \uparrow N) \downarrow S}\ Res_{discont}$$

- Monotony:

$$\frac{A \Rightarrow C}{A \odot B \Rightarrow C \odot B}$$

  Other similar results can be checked.

- Composition:

$$\overline{(C \uparrow B) \odot (B \uparrow A) \Rightarrow C \uparrow A}$$

  Other similar results can be checked.

In categorial grammar (in its type-logical version or in CCG), quantifiers like **everyone** usually have two type assignments, the subject oriented type $S/(N\backslash S)$ and the object oriented type $(S/N)\backslash S$. In our syntactic 1-discontinuous calculus we can assign a unique type $(S \uparrow N) \downarrow S$ which works for both cases, for the following sequents hold:

$$(S \uparrow N) \downarrow S \Rightarrow S/(N\backslash S)$$
$$(S \uparrow N) \downarrow S \Rightarrow (S/N)\backslash S$$

Let's see the first type inference. By the split-wrap rule and residuation $J \bullet (N\backslash S) \Rightarrow S \uparrow N$. By monotony, $(J \bullet (N\backslash S)) \odot ((S \uparrow N) \downarrow S) \Rightarrow (S \uparrow N) \odot ((S \uparrow N) \downarrow S)$. By modus ponens, $(S \uparrow N) \odot ((S \uparrow N) \downarrow S) \Rightarrow S$. By Cut, we have then $(J \bullet (N\backslash S)) \odot ((S \uparrow N) \downarrow S) \Rightarrow S$. By the split-wrap rule, $((S \uparrow N) \downarrow S) \bullet (N\backslash S) \Rightarrow (J \bullet (N\backslash S)) \odot ((S \uparrow N) \downarrow S)$. By Cut again, $(S \uparrow N) \downarrow S \bullet (N\backslash S) \Rightarrow S$. Finally, by residuation, $(S \uparrow N) \downarrow S \Rightarrow S/(N\backslash S)$. The other type inference is completely symmetrical. Later, via the translation τ to 1-**DLC**, we give an extremely easy proof of the type computations we have seen.

Before going on to the following section, we formulate some important questions:

- Is the categorical 1-discontinuous syntactic calculus decidable? Given a sequent $A \Rightarrow B$, is there an effective procedure to check its theoremhood?

- Which kind of prosodic interpretations can we give that make the calculus sound and complete?

The answers to these questions are addressed in the next section.

1.3 From sequents to hypersequents

We define a natural translation of the categorical 1-discontinuous syntactic calculus to a new calculus called hypersequent calculus. In fact, a variety of hypersequent calculi are presented. The idea is quite simple. If a type A is of sort 0, then translate it to A. If A is of sort 1, then translate it to $\sqrt[0]{A}, [\,], \sqrt[1]{A}$. The intuition is that an inhabitant of a sort 1 type A has two components as a 2-dimensional vector[2]. Let's give the intuition of the translation through some examples:

- If B and A are of sort 0, then $(B \uparrow A) \odot A \Rightarrow B$ translates to $\sqrt[0]{B \uparrow A}, A, \sqrt[1]{B \uparrow A} \Rightarrow B$.

- If B and A are of sort 0, then $J \bullet (A \backslash B) \Rightarrow B \uparrow A$ translates to $[\,], A \backslash B \Rightarrow \sqrt[0]{B \uparrow A}, [\,], \sqrt[1]{B \uparrow A}$.

- If A and B are of sort 1, then $A \odot (A \downarrow B) \Rightarrow B$ translates to $\sqrt[0]{A}, \sqrt[0]{A \downarrow B}, [\,], \sqrt[1]{A \downarrow B}, \sqrt[1]{A} \Rightarrow \sqrt[0]{B}, [\,], \sqrt[1]{B}$.

Let's define the translation τ from sequents to hypersequents formally:

[2] Why $\sqrt[0]{\,}, \sqrt[1]{\,}$? Intuition: $\sqrt{A}\sqrt{A} = A$ gives the idea that A (of sort 1) has two components which being concatenated (multiplied!) give A ($A = \sqrt[0]{A} \bullet J \bullet \sqrt[1]{A}$).

$\tau(A \Rightarrow B) := \tau^{-}(A) \Rightarrow \tau^{+}(B)$

$\tau^{-}(I) = \Lambda$

$\tau^{-}(A) = A,$ if A is of sort 0, and the main type constructor of A is different from $\bullet$ *and* $\odot$.

$\tau^{-}(A) = \pi_0(A), [\,], \pi_1(A)$ if A is of sort 1, and the main type constructor of A is different from $\bullet$ *and* $\odot$.

$\tau^{-}(A) = \pi_0(A), [\,], \pi_1(A)$ if A is of sort 1

$\tau^{-}(A \bullet B) = \tau^{-}(A), \tau^{-}(B)$

$\tau^{-}(A \odot B) = \pi_0(A), \tau^{-}(B), \pi_1(A)$

$\pi_0(J) = \Lambda$

$\pi_1(J) = \Lambda$

$\tau^{+}(A) = A$ if A is of sort 0

$\tau^{+}(A) = \sqrt[0]{A}, [\,], \sqrt[1]{A}$ if A is of sort 1

$\pi_0(A \bullet B^1) = A, \pi_0(B^1)$ if A is of sort 0 and B^1 of sort 1

$\pi_1(A \bullet B^1) = \pi_1(B^1)$ if A is of sort 0 *and* B^1 of sort 1

$\pi_0(A^1 \bullet B) = \pi_0(A^1)$ if A^1 is of sort 1 and B of sort 0

$\pi_1(A^1 \bullet B) = \pi_1(A^1), B$ if A^1 is of sort 1 and B of sort 0

$\pi_0(A \odot B) = \pi_0(A), \pi_0(B)$ if both A and B are of sort 1

$\pi_1(A \odot B) = \pi_1(B), \pi_1(A)$ if both A and B are of sort 1

$\pi_0(A) = \sqrt[0]{A}$ if A is of sort 1, different from J and its main type constructor is different from $\bullet$ and $\odot$

$\pi_1(A) = \sqrt[1]{A}$ if A is of sort 1, different from J and its main type constructor is different from $\bullet$ and $\odot$

Example 1. *Let's see the translation of* $((B \uparrow A) \bullet C) \odot A \Rightarrow B \bullet C$*:*

$$\tau(((B \uparrow A) \bullet C) \odot A \Rightarrow B \bullet C) = \tau^{-}(((B \uparrow A) \bullet C) \odot A) \Rightarrow \tau^{+}(B \bullet C)$$

$$\tau^{-}(((B \uparrow A) \bullet C) \odot A) = \pi_0((B \uparrow A) \bullet C),\ A,\ \pi_1((B \uparrow A) \bullet C) =$$

$$\pi_0(B \uparrow A), A,\ \pi_1(B \uparrow A),\ C = \sqrt[0]{B \uparrow A}, A,\ \sqrt[1]{B \uparrow A},\ C$$

$$\tau^{+}(B \bullet C) = B \bullet C$$

We get then:

$$\tau(((B \uparrow A) \bullet C) \odot A \Rightarrow B \bullet C) = \sqrt[0]{B \uparrow A}, A,\ \sqrt[1]{B \uparrow A},\ C \Rightarrow B \bullet C$$

In the categorical calculus, it's easy to see that $((B \uparrow A) \bullet C) \odot A \Leftrightarrow ((B \uparrow A) \odot A) \bullet C$*. Observe that:*

$\tau^{-}(((B \uparrow A) \bullet C) \odot A) = \tau^{-}(((B \uparrow A) \odot A) \bullet C) =$
$\sqrt[0]{B \uparrow A},\ A, \sqrt[1]{B \uparrow A}, C$

For:

$\tau^{-}(((B \uparrow A) \odot A) \bullet C) =$
$\tau^{-}((B \uparrow A) \odot A), \tau^{-}(C) = \pi_0(B \uparrow A), \tau^{-}(A), \pi_1(B \uparrow A), \tau^{-}(C) =$
$\pi_0(B \uparrow A),\ A, \pi_1(B \uparrow A), C =$
$\sqrt[0]{B \uparrow A},\ A, \sqrt[1]{B \uparrow A}, C$

We realize then that the translation τ collapses structural postulates into the same textual form. Structural rules (except an implicit associativity) disappear in the hypersequent calculus, which we can qualify as a pure calculus without structural rules.

Let's define the set of correct configurations of hypersequents (by an unambiguous grammar)[3]:

$\mathcal{O}_0 ::= \Lambda \mid A_0, \mathcal{O}_0, \mid \sqrt[0]{A_1}, \mathcal{O}_0, \sqrt[1]{A_1}, \mathcal{O}_0$
$\mathcal{O}_1 ::= [\,], \mathcal{O}_0 \mid A_0, \mathcal{O}_1 \mid \sqrt[0]{A_1}, \mathcal{O}_1, \sqrt[1]{A_1}\ \mathcal{O}_0 \mid \sqrt[0]{A_1}, \mathcal{O}_0, \sqrt[1]{A_1}\ \mathcal{O}_1$
$\mathcal{O} = \mathcal{O}_0 \mid \mathcal{O}_1$

Remark 4. $[\,]$ *is called the separator, or point of discontinuity.*

We give a pure logical hypersequent calculus, with no structural rules[4]. An important convention is the *vectorial notation*([MFV07]), which consists of writing $\overrightarrow{A}$ for a given type. This notation allows to read A as a sort 0 type or as a sort 1 type with its components. The sort of $\overrightarrow{A}$ is inferred from context. Vectorial notation allows a more compact representation of hypersequents. Figure 1.2 shows the sorted continuous connectives. Figure 1.3 shows the sorted discontinuous product and figure 1.4, the sorted discontinuous implicative connectives. Figure 1.6 shows the rules for the unary operators bridge and split $\hat{}$, $\check{}$ defined in section 1.2 (**1-DLC**$\hat{}\check{}$).

[3] A_0 and A_1 denote arbitrary types of sort respectively 0 and 1.
[4] Except an implicit associativity.

Finally, figure 1.7 shows the Cut rules for the hypersequent calculus. Intuitively, the Cut rule says that $\Rightarrow$ is transitive.

Lemma 2. *Let A, B be arbitrary types. Then,*

$$\vdash A \Rightarrow B \text{ iff } \vdash_{\mathbf{1-DLC}\epsilon\mathbf{+Cut}} \tau(A \Rightarrow B)$$

Proof. We follow again Lambek [58]. In both sides ('*if*' and '*only if*') assume the lemma holds of the premises of the rule. The residuated rules mimic the cases treated by Lambek. Let's the see how the different structural rules of the categorical calculus are mapped through τ, $\sqrt[0]{}$ and $\sqrt[1]{}$. In the following, A, C denote sort 0 types, and B a type of sort 0, or 1.

- Split-wrap rule,

$$\begin{aligned}
&\tau((A \bullet J \bullet C) \odot B) = \tau((A \bullet (J \bullet C)) \odot B) = \\
&\sqrt[0]{A \bullet (J \bullet C)}, \tau(B), \sqrt[1]{A \bullet (J \bullet C)} = \\
&A, \sqrt[0]{J \bullet C}, \tau(B), \sqrt[1]{J \bullet C} = \\
&A, \sqrt[0]{J}, \tau(B), \sqrt[1]{J}, C = \\
&A, \tau(B), C = \\
&\tau(A \bullet B \bullet C)
\end{aligned}$$

Other structural rules behave similarly, that is, they are translated to the same textual form. Finally, the Cut rule[5] is generalized to the cases of the hypersequent calculus using the rules of monotony. $\square$

- **Checking theoremhood of sequents**: So, if we are given a sequent of the categorical 1-discontinuous syntactic calculus, in order to check its theoremhood, we translate it via τ to 1-**DLC**ϵ with Cut, and then we try to prove it in the hypersequent calculus. We will see in the next chapter the positive answer to the problem of the decidability of the categorical 1-discontinuous syntactic calculus. The 1-**DLC**ϵ with Cut is decidable.

Remark 5. *Figure 1.5 presents a surprising fact. According to the intuitive prosodic interpretation, the discontinuous unit J could be definable in terms of the continuous*

[5] Actually, there are two instances of Cut in the hypersequent calculus (for sort 0 or 1).

$$\frac{}{\overrightarrow{A} \Rightarrow \overrightarrow{A}}\; id$$

$$\frac{\Delta_1, A, B, \Delta_2 \Rightarrow \overrightarrow{D}}{\Delta_1, A \bullet B, \Delta_2 \Rightarrow \overrightarrow{D}}\; \bullet L \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Delta, \Gamma \Rightarrow A \bullet B}\; \bullet R$$

$$\frac{\Delta_1, \sqrt[0]{A}, \Theta, \sqrt[1]{A}, B, \Delta_2 \Rightarrow \overrightarrow{D}}{\Delta_1, \sqrt[0]{A \bullet B}, \Theta, \sqrt[1]{A \bullet B}, \Delta_2 \Rightarrow \overrightarrow{D}}\; \bullet L \qquad \frac{\Gamma \Rightarrow \sqrt[0]{A}, [\,], \sqrt[1]{A} \quad \Delta \Rightarrow B}{\Delta, \Gamma \Rightarrow \sqrt[0]{A \bullet B}, [\,], \sqrt[1]{A \bullet B}}\; \bullet R$$

$$\frac{\Delta_1, A, \sqrt[0]{B}, \Theta, \sqrt[1]{B}, \Delta_2 \Rightarrow \overrightarrow{D}}{\Delta_1, \sqrt[0]{A \bullet B}, \Theta, \sqrt[1]{A \bullet B}, \Delta_2 \Rightarrow \overrightarrow{D}}\; \bullet L \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow \sqrt[0]{B}, [\,], \sqrt[1]{B}}{\Delta, \Gamma \Rightarrow \sqrt[0]{A \bullet B}, [\,], \sqrt[1]{A \bullet B}}\; \bullet R$$

$$\frac{\Gamma \Rightarrow A \quad \Delta_1, C, \Delta_2 \Rightarrow \overrightarrow{D}}{\Delta_1, \Gamma, A\backslash C, \Delta_2 \Rightarrow \overrightarrow{D}}\; \backslash L \qquad \frac{A, \Gamma \Rightarrow C}{\Gamma \Rightarrow A\backslash C}\; \backslash R$$

$$\frac{\Gamma_1, [\,], \Gamma_2 \Rightarrow \sqrt[0]{A}, [\,], \sqrt[1]{A} \quad \Delta_1, \sqrt[0]{C}, \Theta, \sqrt[1]{C}, \Delta_2 \Rightarrow \overrightarrow{D}}{\Delta_1, \Gamma_1, \Theta, \Gamma_2, A\backslash C, \Delta_2 \Rightarrow \overrightarrow{D}}\; \backslash L \qquad \frac{\sqrt[0]{A}, [\,], \sqrt[1]{A}, \Gamma \Rightarrow \sqrt[0]{C}, [\,], \sqrt[1]{C}}{\Gamma \Rightarrow A\backslash C}\; \backslash R$$

$$\frac{\Gamma \Rightarrow A \quad \Delta_1, \sqrt[0]{C}, \Theta, \sqrt[1]{C}, \Delta_2 \Rightarrow \overrightarrow{D}}{\Delta_1, \Gamma, \sqrt[0]{A\backslash C}, \Theta, \sqrt[1]{A\backslash C}, \Delta_2 \Rightarrow \overrightarrow{D}}\; \backslash L \qquad \frac{A, \Gamma \Rightarrow \sqrt[0]{C}, [\,], \sqrt[1]{C}}{\Gamma \Rightarrow \sqrt[0]{A\backslash C}, [\,], \sqrt[0]{A\backslash C}}\; \backslash R$$

$$\frac{\Gamma \Rightarrow A \quad \Delta_1, C, \Delta_2 \Rightarrow \overrightarrow{D}}{\Delta_1, C/A, \Gamma, \Delta_2 \Rightarrow \overrightarrow{D}}\; /L \qquad \frac{\Gamma, A \Rightarrow C}{\Gamma \Rightarrow C/A}\; /R$$

$$\frac{\Gamma 1, [\,], \Gamma 2 \Rightarrow \sqrt[0]{A}, [\,], \sqrt[1]{A} \quad \Delta_1, \sqrt[0]{C}, \Theta, \sqrt[1]{C}, \Delta_2 \Rightarrow \overrightarrow{D}}{\Delta_1, C/A, \Gamma_1, \Theta, \Gamma_2, \Delta_2 \Rightarrow \overrightarrow{D}}\; /L \qquad \frac{\Gamma, \sqrt[0]{A}, [\,], \sqrt[1]{A} \Rightarrow \sqrt[0]{C}, [\,], \sqrt[1]{C}}{\Gamma \Rightarrow C/A}\; /R$$

$$\frac{\Gamma \Rightarrow A \quad \Delta_1, \sqrt[0]{C}, \Theta, \sqrt[1]{C}, \Delta_2 \Rightarrow \overrightarrow{D}}{\Delta_1, \sqrt[0]{C/A}, \Theta, \sqrt[1]{C/A}, \Gamma, \Delta_2 \Rightarrow \overrightarrow{D}}\; /L \qquad \frac{\Gamma, A \Rightarrow \sqrt[0]{C}, [\,], \sqrt[1]{C}}{\Gamma \Rightarrow \sqrt[0]{C/A}, [\,], \sqrt[0]{C/A}}\; /R$$

Figure 1.2: Axioms and rules for continuous connectives of the hypersequent calculus **1-DLC**

$$\frac{\Delta_1, \sqrt[0]{A}, B, \sqrt[1]{A}, \Delta_2 \Rightarrow \overrightarrow{D}}{\Delta_1, A \odot B, \Delta_2 \Rightarrow \overrightarrow{D}} \odot L \qquad \frac{\Gamma_1, [\,], \Gamma_2 \Rightarrow \sqrt[0]{A}, [\,], \sqrt[1]{A} \quad \Delta \Rightarrow B}{\Gamma_1, \Delta, \Gamma_2 \Rightarrow A \odot B} \odot R$$

$$\frac{\Delta_1, \sqrt[0]{A}, \sqrt[0]{B}, \Delta_2, \sqrt[1]{B}, \sqrt[1]{A}, \Delta_3 \Rightarrow \overrightarrow{D}}{\Delta_1, \sqrt[0]{A \odot B}, \Delta_2, \sqrt[1]{A \odot B}, \Delta_3 \Rightarrow \overrightarrow{D}} \odot L \quad \frac{\Gamma_1, [\,], \Gamma_2 \Rightarrow \sqrt[0]{A}, [\,], \sqrt[1]{A} \quad \Delta_1, [\,], \Delta_2 \Rightarrow \sqrt[0]{B}, [\,], \sqrt[1]{B}}{\Gamma_1, \Delta_1, [\,], \Delta_2, \Gamma_2 \Rightarrow \sqrt[0]{A \odot B}, [\,], \sqrt[1]{A \odot B}} \odot R$$

Figure 1.3: Rules for the discontinuous product of the hypersequent calculus **1-DLC**

$$\frac{\Gamma_1, [\,], \Gamma_2 \Rightarrow \sqrt[0]{A}, [\,], \sqrt[1]{A} \quad \Delta_1, C, \Delta_2 \Rightarrow \overrightarrow{D}}{\Delta_1, \Gamma_1, A \downarrow C, \Gamma_2, \Delta_2 \Rightarrow \overrightarrow{D}} \downarrow L \qquad \frac{\sqrt[0]{A}, \Gamma, \sqrt[1]{A} \Rightarrow C}{\Gamma \Rightarrow A \downarrow C} \downarrow R$$

$$\frac{\Gamma_1, [\,], \Gamma_2 \Rightarrow \sqrt[0]{A}, [\,], \sqrt[1]{A} \quad \Delta_1, \sqrt[0]{C}, \Delta_2, \sqrt[1]{C}, \Delta_3 \Rightarrow \overrightarrow{D}}{\Delta_1, \Gamma_1, \sqrt[0]{A \downarrow B}, \Delta_2, \sqrt[1]{A \downarrow B}, \Gamma_2 \Rightarrow \overrightarrow{D}} \downarrow L \quad \frac{\sqrt[0]{A}, \Gamma, \sqrt[1]{A} \Rightarrow \sqrt[0]{B}, [\,], \sqrt[1]{B}}{\Gamma \Rightarrow \sqrt[0]{A \downarrow B}, [\,], \sqrt[1]{A \downarrow B}} \downarrow R$$

$$\frac{\Gamma \Rightarrow B \quad \Delta_1, C, \Delta_2 \Rightarrow D}{\Delta_1, \sqrt[0]{C \uparrow B}, \Gamma, \sqrt[1]{C \uparrow B}, \Delta_2 \Rightarrow D} \uparrow L \qquad \frac{\Gamma_1, B, \Gamma_2 \Rightarrow C}{\Gamma_1, [\,], \Gamma_2 \Rightarrow \sqrt[0]{C}, [\,], \sqrt[1]{C}} \uparrow R$$

$$\frac{\Gamma_1, [\,], \Gamma_2 \Rightarrow \sqrt[0]{B}, [\,], \sqrt[1]{B} \quad \Delta_1, \sqrt[0]{C}, \Delta_2, \sqrt[1]{C}, \Delta_3 \Rightarrow D}{\Delta_1, \sqrt[0]{C \uparrow B}, \Gamma_1, \Delta_2, \Gamma_2, \sqrt[1]{C \uparrow B}, \Delta_3 \Rightarrow D} \uparrow L \quad \frac{\Gamma_1, \sqrt[0]{B}, [\,], \sqrt[0]{B}, \Gamma_2 \Rightarrow C}{\Gamma_1, [\,], \Gamma_2 \Rightarrow \sqrt[0]{C \uparrow B}, [\,], \sqrt[1]{C \uparrow B}} \uparrow R$$

Figure 1.4: Rules for the discontinuous implicative connectives of the hypersequent calculus **1-DLC**

$$\frac{\Delta, \Gamma \Rightarrow \overrightarrow{D}}{\Delta, I, \Gamma \Rightarrow \overrightarrow{D}} I\,L \qquad \frac{}{\Rightarrow I} I\,R$$

$$\frac{\Delta_1, \Delta_2, \Delta_3 \Rightarrow \overrightarrow{D}}{\Delta_1, \sqrt[0]{J}, \Delta_2, \sqrt[1]{J}, \Delta_3 \Rightarrow \overrightarrow{D}} J\,L \qquad \frac{}{[\,] \Rightarrow \sqrt[0]{J}, [\,], \sqrt[1]{J}} J\,R$$

Figure 1.5: Continuous and discontinuous rules for the units of **1-DLC**

$$\frac{\Gamma_1, \sqrt[0]{A}, \sqrt[1]{A}, \Gamma_2 \Rightarrow C}{\Gamma_1, \hat{}A, \Gamma_2 \Rightarrow C} \hat{}\,L \qquad \frac{\Gamma_1, [\,], \Gamma_2 \Rightarrow \sqrt[0]{A}, [\,], \sqrt[1]{A}}{\Gamma_1, \Gamma_2 \Rightarrow \hat{}A} \hat{}\,R$$

$$\frac{\Gamma_1, A, \Gamma_2 \Rightarrow C}{\Gamma_1, \sqrt[0]{\check{}A}, \sqrt[1]{\check{}A}, \Gamma_2 \Rightarrow C} \check{}\,L \qquad \frac{\Gamma_1, \Gamma_2 \Rightarrow B}{\Gamma_1, [\,], \Gamma_2 \Rightarrow \sqrt[0]{\check{}B}, [\,], \sqrt[1]{\check{}B}} \check{}\,R$$

Figure 1.6: Rules for the unary connectives $\hat{}$ and $\check{}$

$$\frac{\Gamma \Rightarrow A \quad \Delta_1, A, \Delta_2 \Rightarrow \overrightarrow{D}}{\Delta_1, \Gamma, \Delta_2 \Rightarrow \overrightarrow{D}}\ \mathbf{Cut_0} \qquad \frac{\Gamma_1, [\,], \Gamma_2 \Rightarrow \sqrt[0]{A}, [\,], \sqrt[1]{A} \quad \Delta_1, \sqrt[0]{A}, \Delta_2, \sqrt[1]{A}, \Delta_3 \Rightarrow \overrightarrow{D}}{\Delta_1, \Gamma_1, \Delta_2, \Gamma_2, \Delta_3 \Rightarrow \overrightarrow{D}}\ \mathbf{Cut_1}$$

Figure 1.7: Cut rules for hypersequent calculus

unit I, i.e, $\mathbb{I} \Uparrow \mathbb{I} = \mathbb{J} = \{I \cdot \$ \cdot I\} = \{\$\}$. We would expect then the left rule of the discontinuous unit to be derivable. This is not the case. In chapter 3, we will see the problems (namely incompleteness) we face with a definable discontinuous unit.

Definition 7. *A provable hypersequent $\mathcal{S} = \Delta \Rightarrow \overrightarrow{A}$ is the end hypersequent of a (finite) derivation in 1-**DLC**, or 1-**DLC**ϵ, that is, 1-**DLC** with units. We write $\vdash \Delta \Rightarrow \overrightarrow{A}$.*

Remark 6. *Within 1-**DLC**, or 1-**DLC**ϵ, there are a variety of subcalculi depending on the sort. Morrill and Fadda [2005] have an important contribution to a subcalculus of 1-**DLC**, or 1-**DLC**ϵ, which they call* Basic Discontinuity Calculus. *In that work, they present the hypersequent calculus restricted to the basic continuous case and the discontinous connectives $\odot, \uparrow, \downarrow$ of sort functionality respectively $(1,0) \to 0, (0,0) \to 1, (1,0) \to 0$, without units.*

We saw before the derivation $(S{\uparrow}N){\downarrow}S \Rightarrow S/(N\backslash S)$ in the categorical 1-discontinuous calculus. The translation of this sequent to the hypersequent calculus has the same textual form. Let's see its proof in 1-**DLC**:

$$\cfrac{\cfrac{\cfrac{\cfrac{\overline{N \Rightarrow N} \quad \overline{S \Rightarrow S}}{N, N\backslash S \Rightarrow S}\ \backslash L}{[\,], N\backslash S \Rightarrow S \uparrow N}\ \uparrow R \quad \overline{S \Rightarrow S}}{(S{\uparrow}N){\downarrow}S, N\backslash S \Rightarrow S}\ \downarrow L}{(S{\uparrow}N){\downarrow}S \Rightarrow S/(N\backslash S)}\ /R$$

We give now the derivation of the discontinuous quantifier type which entails the object-oriented continuous quantifier type:

$$\cfrac{\cfrac{\cfrac{\vdots}{S/N, [\,] \Rightarrow S \uparrow N}\ \uparrow R \quad \overline{S \Rightarrow S}}{S/N, (S{\uparrow}N){\downarrow}S \Rightarrow S}\ \downarrow L}{(S{\uparrow}N){\downarrow}S \Rightarrow (S/N)\backslash S}\ \backslash R$$

We will see more linguistic examples in chaper 4.

Finally, we give an answer to the problem of the prosodic interpretation of the categorical and hypersequent calculi presented in this chapter:

- **Is there a (prosodic) interpretation of the 1-DLC (and thus of the categorical calculus) which is sound and complete**?

The answer is again positive. The standard prosodic interpretation in 1-graded monoids we've seen in this chapter, is nearly the right one (see chapter 3).

1.4 Conclusions

We have seen a categorical calculus which is the basis of a type-logical account of discontinuity. This calculus has been driven model-theoretically. The problems of prosodic interpretation (soundness and completeness) and decidability have a satisfactory answer through the translation to the hypersequent calculus 1-**DLC**ϵ. This calculus is a natural extension of the (continuous) Lambek calculus which is a pure substructural logic without structural rules (except associativity). 1-**DLC**ϵ, like the continuous Lambek calculus LC, is a pure substructural logic without structural rules (except associativity). In the next chapters, we study the proof theory and semantic interpretation of 1-**DLC** with or without units, which as we know, contains several subcalculi depending on the set of types we work with (according to the sort).

Chapter 2

Proof theory

We present some proof-theoretical results of the hypersequent calculus. The main result is Cut-elimination for the the 1-**DLC**,1-**DLC**ϵ and **1-DLC**$\hat{}\check{}$, which is the hypersequent calculus with two extra unary logical operators invented by Morrill and Merenciano [MM96] (see chapter 1). These connectives, called *bridge* and *split* are very elegant as they give a way to increment or decrement the sort of a type. Other results cover results on invertible rules, the subformula property and as a consequence, the decidability of 1-**DLC** and 1-**DLC**ϵ (and **1-DLC**$\hat{}\check{}$). This gives the utility of the translation τ of the 1-discontinuous categorical syntactic calculus to the hypersequent calculus.

2.1 Technical preliminaries

2.1.1 On weight of types and configurations

Definition 8 (weight of types)**.** *If A is a type of sort 0 or 1,*

$$w(A) = \begin{cases} w(A) = 0 & \textit{if } A \textit{ is atomic} \\ w(I) = 1 & \\ w(J) = 1 & \\ w(A * B) = w(A) + w(B) + 1 & \textit{for a complex type, where } * = \backslash, /, \bullet, \downarrow, \uparrow, \odot \end{cases}$$

Remark 7. *Let $\Delta_1, \sqrt[0]{A}, \Delta_2, \sqrt[1]{A}, \Delta_3$ be the antecedent of a hypersequent. According to the definition of the configurations (i.e., the antecedents of a hypersequent) given in the last chapter, the meta-variables of Δ_i may be (correct) configurations or not! Consider $\sqrt[0]{C}, B/A, A, \sqrt[1]{C}$ which is a correct configuration. Then, $\sqrt[0]{C}$ and $\sqrt[1]{C}$ are not correct configurations. Our goal is to define the weight of an antecedent configuration.*

Definition 9. *Let $A_1, \cdots, A_n$ be an antecedent configuration with no occurrences of the components $\sqrt[0]{}, \sqrt[1]{}$ of a sort 1 type. Then:*

$$w(A_1, \cdots, A_n) = w(A_1) + \cdots + w(A_n)$$

The separator doesn't contribute to the weight of a configuration, so:

$$w([\,]) = 0$$

Consider now the configuration $\sqrt[0]{C}, \Delta, \sqrt[1]{C}$ with Δ free of $\sqrt[0]{}, \sqrt[1]{}$. Then:

$$w(\sqrt[0]{C}, \Delta, \sqrt[1]{C}) = w(C) + w(\Delta)$$

Lemma 3. *Let Δ be a correct antecedent configuration of a hypersequent. Let $Typ(\Delta)$ be the multiset of types occurring in Δ. Then:*

$$w(\Delta) = \sum_{sort(A)=0,\ A\in Typ(\Delta)} w(A) + \sum_{sort(A)=1,\ A\in Typ(\Delta)} w(A)$$

Proof. Induction on the recursive definition of configurations. □

2.1.2 A compact representation of hypersequent calculus

In order to make less painful the proof of cut-elimination, we present a more compact representation of hypersequent calculus:

If we have $\Delta_1, \sqrt[0]{A}, \Delta_2, \sqrt[1]{A}, \Delta_3 \Rightarrow \overrightarrow{C}$, *we write instead* $\Delta_1, \overrightarrow{A}(\Delta_2), \Delta_3 \Rightarrow \overrightarrow{C}$. *If* Δ *is a configuration, we can write* $\overrightarrow{\Delta}(\overrightarrow{B})$, *which has the obvious meaning, i.e.,* Δ *wraps* B. *If* Δ *is of sort 0, then* $\overrightarrow{\Delta}(\overrightarrow{B})$ *must be read* Δ. *Vectorial notation may be nested. For example, in* $\overrightarrow{\Delta, \overrightarrow{A\backslash C}}$ *it must be understood that* $\Delta, A\backslash C$ *is split around the separator* $[\,]$. *The nested vector arrows of types or configurations have the meaning that they may be of sort 0 or sort 1. If* Δ *is a configuration of sort 0, then* $\overrightarrow{\Delta}(\Theta)$ *must be read* Δ *with* Θ *empty, because* Δ *cannot wrap anything. So, the reader shouldn't have problems with the weight of vectorial configurations, e.g.* $w(\overrightarrow{\Delta}(\Theta))$. *Depending on the sort of* Δ, *the weight is computed as before.*

2.2 Cut-elimination

We consider a new sorted rule, called the cut rule, *for the hypersequent calculus. See figure 1.7. The idea behind the cut rule is the transitivity of* $\Rightarrow$. *The main goal of this section is to show that this rule is admissible in the hypersequent calculus without cut.*

Definition 10 (cut formula)**.** *The type* A *which appears twice in the cut rule, namely in the succedent of the left premise and in the antecedent of the right premise of cut is called the* ***cut formula***.

Remark 8. *It's important that the reader notices that in fact this rule has two instances according to the sort of the cut formula. Again, the compact representation*

$$\frac{\Delta \Rightarrow \overrightarrow{A} \quad \Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}\ cut$$

Figure 2.1: Cut rule for **DLC** in compact notation

of hypersequent calculus is used.

Definition 11 (cut degree)**.** *Given a hypersequent calculus derivation $\mathcal{D}$ whose last rule is the* cut rule*, and assuming that its premises have been proved without the cut rule, we define the* cut degree *of the derivation as:*

$$d(\mathcal{D}) = w(A) + w(D) + w(\Delta) + w(\Phi_0, \Phi_2)$$

We prove now the cut elimination theorem, or admissibility of the cut rule.

Theorem 1 (cut elimination for the discontinuous calculi)**.** *The cut rule is admissible for the calculi,* ***1*-DLC***,* ***1*-DLC**ϵ*,* ***1*-DLC**$^{\hat{}\check{}}$*. This means that for any proof derivation $\mathcal{D}$ of a given sequent $\mathcal{S}$ in any of these calculi with the additional rule* ***cut****, there is a cut-free derivation $\mathcal{D}^*$ whose conclusion is the sequent $\mathcal{S}$.*

Proof. **Main idea of the proof**:
The strategy of the proof follows Lambek's proof [58]. We assume we have a hypersequent calculus derivation $\mathcal{D}$ whose last rule is *cut*. The premises of the cut rule are supposed to be cut-free derived. $\mathcal{D}$ has a cut degree d. We give a procedure (effective) which transforms the derivation into a new one $\mathcal{D}^*$ with the same hypersequent as conclusion but whose cut degree d^* is smaller than d. We repeat this procedure until eventually we get a proof derivation with cut degree 0, i.e., the final derivation is cut-free. The number of transformations is obviously finite[1]. So, the procedure terminates.

The proof has the following structure:

i) Axiom case in the premises.

[1]Because of the Least Number Principle of ω: All strictly decreasing sequences of natural numbers are finite.

ii) Permutation conversions: The active formula (introduced by a rule $\mathcal{R}$) of one of the premises of cut is different from the cut formula. In this case, the transformation is simply to apply the cut rule before $\mathcal{R}$. We permute the order of application of the rules. We will check that the resulting cut degree is smaller.

iii) Principal cut: The active formula of both premises of cut (of cut degree d) are the cut formula. The transformation here consists of applying the cut rule twice, with degrees smaller than d.

Remark 9. *All connectives involved in the proof are sort polymorphic.*

- ***Axiom case***

  One of the Cut premises is an axiom of sort 0 or sort 1:

  Suppose the left premise is the axiom:

$$\frac{\overrightarrow{A} \Rightarrow \overrightarrow{A} \quad \Delta_1,\ \overrightarrow{A}(\Delta_2), \Delta_3 \Rightarrow \overrightarrow{C}}{\Delta_1,\ \overrightarrow{A}(\Delta_2), \Delta_3 \Rightarrow \overrightarrow{C}}$$

  The inference is reduced to the right premise of the cut. Analogously, for the symmetric case.

- ***Permutation conversion cases***

  The cut is not principal. Then the cut formula is not active in one of the premises.

  - The cut formula is not active in the left premise of the cut:

  - The left premise of cut is $\bullet L$:

$$\mathcal{D} = \frac{\dfrac{\Delta_0, B, \overrightarrow{C}(\Delta_1), \Delta_2 \Rightarrow \overrightarrow{A}}{\Delta_0, \overrightarrow{B \bullet C}(\Delta_1), \Delta_2 \Rightarrow \overrightarrow{A}}\, \bullet L \qquad \Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}{\Phi_0, \overrightarrow{\Delta_0, \overrightarrow{B \bullet C}(\Delta_1), \Delta_2}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}\, cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* \qquad \dfrac{\dfrac{\Delta_0, B, \overrightarrow{C}(\Delta_1), \Delta_2 \Rightarrow \overrightarrow{A} \quad \Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}{\Phi_0, \overrightarrow{\Delta_0, B, \overrightarrow{C}(\Delta_1), \Delta_2}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}}{\Phi_0, \overrightarrow{\Delta_0, \overrightarrow{B \bullet C}(\Delta_1), \Delta_2}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}$$

Since $w(B)+w(C) < w(B\bullet C)$, $w(\overrightarrow{\Delta_0, B, \overrightarrow{C}(\Delta_1), \Delta_2}(\Phi_1)) < w(\overrightarrow{\Delta_0, \overrightarrow{B \bullet C}(\Delta_1), \Delta_2}(\Phi_1))$, *whence* $d(\mathcal{D}) < d(\mathcal{D}^*)$.

- The left premise of cut is $\backslash L$*:*

$$\mathcal{D} = \dfrac{\dfrac{\Gamma \Rightarrow \overrightarrow{B} \quad \Delta_0, \overrightarrow{C}(\Delta_1), \Delta_2 \Rightarrow \overrightarrow{A}}{\Delta_0, \overrightarrow{\Gamma, B\backslash C}(\Delta_1), \Delta_2 \Rightarrow \overrightarrow{A}}\,\backslash L \qquad \Phi_0\, \overrightarrow{A}(\Phi_1), \Phi_2 \Rightarrow D}{\Phi_0, \overrightarrow{\Delta_0, \overrightarrow{\Gamma, B\backslash C}(\Delta_1), \Delta_2}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}\,\mathit{cut}$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \dfrac{\Gamma \Rightarrow \overrightarrow{B} \qquad \dfrac{\Delta_0, \overrightarrow{C}(\Delta_1), \Delta_2 \Rightarrow \overrightarrow{A} \quad \Phi_0\, \overrightarrow{A}(\Phi_1), \Phi_2 \Rightarrow D}{\Phi_0, \overrightarrow{\Delta_0, \overrightarrow{C}(\Delta_1), \Delta_2}(\Phi_1)\Phi_2 \Rightarrow \overrightarrow{D}}\,\mathit{cut}}{\Phi_0, \overrightarrow{\Delta_0, \overrightarrow{\Gamma, B\backslash C}(\Delta_1), \Delta_2}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}\,\backslash L$$

Again some formula material is lost in the earlier cut, so $d(\mathcal{D}) < d(\mathcal{D}^*)$.

The case with rule $/L$ *is completely similar to the case with rule* $\backslash L$.

- The left premise of cut is obtained by rule $\odot L$*:*

$$\mathcal{D} = \dfrac{\dfrac{\Delta_0, \overrightarrow{B}(\overrightarrow{C}(\Delta_1)), \Delta_2 \Rightarrow \overrightarrow{A}}{\Delta_0, \overrightarrow{B \odot C}(\Delta_1), \Delta_2 \Rightarrow \overrightarrow{A}}\,\odot L \qquad \Phi_0, A(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}{\Phi_0, \overrightarrow{\Delta_0, \overrightarrow{B \odot C}(\Delta_1), \Delta_2}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}\,\mathit{cut}$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \quad \dfrac{\dfrac{\Delta_0, B(C(\Delta_1)), \Delta_2 \Rightarrow \overrightarrow{A} \quad \Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}{\Phi_0, \overrightarrow{\Delta_0, B(C(\Delta_1)), \Delta_2}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}\, cut}{\Phi_0, \overrightarrow{\Delta_0, \overrightarrow{B \odot C}(\Delta_1), \Delta_2}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}\, \odot L$$

$d(\mathcal{D}) < d(\mathcal{D}^*)$, for $w(B) + w(C) < w(B \odot C)$.

- The left premise of cut is obtained by rule $\uparrow L$:

$$\mathcal{D} \quad \dfrac{\dfrac{\Gamma \Rightarrow \overrightarrow{B} \quad \Delta_0, C(\Delta_1), \Delta_2 \Rightarrow \overrightarrow{A}}{\Delta_0, \overrightarrow{C \uparrow B}(\overrightarrow{\Gamma}(\Delta_1)), \Delta_2 \Rightarrow \overrightarrow{A}}\, \uparrow L \qquad \Phi_0, A(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}{\Phi_0, \overrightarrow{\Delta_0, \overrightarrow{C \uparrow B}(\overrightarrow{\Gamma}(\Delta_1)), \Delta_2}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}\, cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \quad \dfrac{\Gamma \Rightarrow \overrightarrow{B} \qquad \dfrac{\Delta_0, C(\Delta_1), \Delta_2 \Rightarrow \overrightarrow{A} \quad \Phi_0, A(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}{\Phi_0, \overrightarrow{\Delta_0, \overrightarrow{C}(\Delta_1), \Delta_2}(\Phi_1)\Phi_2 \Rightarrow \overrightarrow{D}}\, cut}{\Phi_0, \overrightarrow{\Delta_0, \overrightarrow{C \uparrow B}(\overrightarrow{\Gamma}(\Delta_1)), \Delta_2}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}\, \uparrow L$$

$d(\mathcal{D}) < d(\mathcal{D}^*)$, for $w(B) + w(C) < w(B \uparrow C)$.

The case involving $\downarrow L$ is completely similar to the case with rule $\uparrow L$.

- The cut formula is not active in the right premise of the cut. In this case, the last rule of the right premise can be anything.

- Suppose the last rule of the right premise is $\backslash L$:

$$\mathcal{D} = \quad \dfrac{\Delta \Rightarrow \overrightarrow{A} \qquad \dfrac{\Phi_1, \overrightarrow{A}(\Phi_2), \Phi_3 \Rightarrow \overrightarrow{B} \quad \Phi_0, \overrightarrow{C}(\Phi_4), \Phi_5 \Rightarrow \overrightarrow{D}}{\Phi_0, \overrightarrow{\Phi_1, \overrightarrow{A}(\Phi_2), \Phi_3, B\backslash C}(\Phi_4), \Phi_5 \Rightarrow \overrightarrow{D}}\, \backslash L}{\Phi_0, \overrightarrow{\Phi_1, \overrightarrow{\Delta}(\Phi_2), \Phi_3, B\backslash C}(\Phi_4), \Phi_5 \Rightarrow \overrightarrow{D}}\, cut$$

$\rightsquigarrow$

$$\mathcal{D}^* = \frac{\dfrac{\Delta \Rightarrow \overrightarrow{A} \quad \Phi_1, \overrightarrow{A}(\Phi_2), \Phi_3 \Rightarrow \overrightarrow{B}}{\Phi_1, \overrightarrow{\Delta}(\Phi_2), \Phi_3 \Rightarrow \overrightarrow{B}}\,cut \qquad \Phi_0, \overrightarrow{C}(\Phi_4), \Phi_5 \Rightarrow \overrightarrow{D}}{\Phi_0, \overrightarrow{\Phi_1, \overrightarrow{\Delta}(\Phi_2), \Phi_3, B\backslash} C(\Phi_4), \Phi_5 \Rightarrow \overrightarrow{D}}\,\backslash L$$

The new derivation has smaller cut degree, for $w(B) + w(C) < w(B\backslash C)$, and then $d(\mathcal{D}*) = w(A) + w(B) + w(\Phi_1, \overrightarrow{\Delta}(\Phi_2), \Phi_3) < d(\mathcal{D}) = w(A) + w(D) + w(\overrightarrow{\Phi_0, \Phi_1, \overrightarrow{\Delta}(\Phi_2)}, \Phi_3, B\backslash C(\Phi_4), \Phi_5)$.

Cases where the cut formula A appear in the right premise of the $\backslash L$ rule are completely similar.

- Suppose the last rule of the right premise is $\backslash R$:

$$\mathcal{D} = \frac{\Delta \Rightarrow A \quad \dfrac{B, \Phi_0, A(\Phi_1), \Phi_2 \Rightarrow C}{\Phi_0, A(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{B\backslash C}}\,\backslash R}{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{B\backslash C}}\,cut$$

$\rightsquigarrow$

$$\mathcal{D}^* = \frac{\dfrac{\Delta \Rightarrow \overrightarrow{A} \quad \overrightarrow{B}, \Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{C}}{B, \Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{C}}\,cut}{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{B\backslash C}}\,\backslash R$$

Again, as expected, $w(B) + w(C) < w(B\backslash C)$ leads to $d(\mathcal{D}^*) < d(\mathcal{D})$.

- Suppose the last rule of the right premise is $\uparrow L$:

$$\mathcal{D} = \frac{\Delta \Rightarrow \overrightarrow{A} \quad \dfrac{\Phi_1, \overrightarrow{A}(\Phi_2), \Phi_3 \Rightarrow \overrightarrow{B} \quad \Phi_0, \overrightarrow{C}(\Phi_4), \Phi_5 \Rightarrow \overrightarrow{D}}{\Phi_0, \Phi_1, \overrightarrow{C \uparrow B}(\overrightarrow{\Phi_1, \overrightarrow{A}(\Phi_2), \Phi_3}(\Phi_4)), \Phi_5 \Rightarrow \overrightarrow{D}}\,\backslash L}{\Phi_0, \Phi_1, \overrightarrow{C \uparrow B}(\overrightarrow{\Phi_1, \overrightarrow{\Delta}(\Phi_2), \Phi_3}(\Phi_4)), \Phi_5 \Rightarrow \overrightarrow{D}}\,cut$$

$\rightsquigarrow$

$$\mathcal{D}^* = \cfrac{\cfrac{\Delta \Rightarrow \overrightarrow{A} \quad \Phi_1, \overrightarrow{A}(\Phi_2), \Phi_3 \Rightarrow \overrightarrow{B}}{\Phi_1, \overrightarrow{\Delta}(\Phi_2), \Phi_3 \Rightarrow \overrightarrow{B}}\, cut \qquad \Phi_0, \overrightarrow{C}(\Phi_4), \Phi_5 \Rightarrow \overrightarrow{D}}{\Phi_0, \Phi_1, \overrightarrow{C \uparrow B}(\overrightarrow{\Phi_1, \overrightarrow{\Delta}(\Phi_2), \Phi_3}(\Phi_4)), \Phi_5 \Rightarrow \overrightarrow{D}}\uparrow L$$

Again, as expected, $w(B) + w(C) < w(B \uparrow C)$ *leads to* $d(\mathcal{D}^*) < d(\mathcal{D})$.

Cases where the cut formula A *appears in the right premise of* $\uparrow L$ *rule are completely similar.*

- Suppose the last rule of the right premise is $\uparrow R$*:*

$$\mathcal{D} = \cfrac{\Delta \Rightarrow \overrightarrow{A} \quad \cfrac{\Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2, \overrightarrow{B}, \Phi_3 \Rightarrow C}{\Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2, [\,], \Phi_3 \Rightarrow \overrightarrow{C \uparrow B}}\uparrow R}{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2, [\,], \Phi_3 \Rightarrow \overrightarrow{C \uparrow B}}\, cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \cfrac{\cfrac{\Delta \Rightarrow \overrightarrow{A} \quad \Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2, \overrightarrow{B}, \Phi_3 \Rightarrow \overrightarrow{C}}{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2, \overrightarrow{B}, \Phi_3 \Rightarrow \overrightarrow{C}}\, cut}{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2, [\,], \Phi_3 \Rightarrow \overrightarrow{C \uparrow B}}\uparrow R$$

Again, $d(\mathcal{D}) < d(\mathcal{D}^*)$*, for* $w(B) + w(C) < w(C \uparrow B)$

- The case in which the last rule of the right premise is $\downarrow L$ *or* $\downarrow R$ *is completely similar to* $\uparrow$.

- The last rule of the right premise is $\bullet L$*:*

$$\mathcal{D} = \cfrac{\Delta \Rightarrow \overrightarrow{A} \quad \cfrac{\Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2, \overrightarrow{B}(\Phi_3), C, \Phi_4 \Rightarrow \overrightarrow{D}}{\Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2, \overrightarrow{B \bullet C}(\Phi_3), \Phi_4 \Rightarrow \overrightarrow{D}}\bullet L}{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2, \overrightarrow{B \bullet C}(\Phi_3), \Phi_4 \Rightarrow \overrightarrow{D}}\, cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \cfrac{\cfrac{\Delta \Rightarrow A \quad \Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2, \overrightarrow{B}(\Phi_3), C, \Phi_4 \Rightarrow \overrightarrow{D}}{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2, \overrightarrow{B}(\Phi_3), C, \Phi_4 \Rightarrow \overrightarrow{D}}\ cut}{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2, \overrightarrow{B \bullet C}(\Phi_3), \Phi_4 \Rightarrow \overrightarrow{D}}\ \bullet L$$

- The last rule of the right premise is $\bullet R$:

$$\mathcal{D} = \cfrac{\Delta \Rightarrow A \quad \cfrac{\Phi_0, A(\Phi_1), \Phi_2 \Rightarrow B \quad \Phi_3 \Rightarrow C}{\Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2, \Phi_3 \Rightarrow \overrightarrow{B \bullet C}}\ \bullet R}{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2, \Phi_3 \Rightarrow \overrightarrow{B \bullet C}}\ cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \cfrac{\cfrac{\Delta \Rightarrow A \quad \Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2 \Rightarrow B}{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2 \Rightarrow B}\ cut \qquad \Phi_3 \Rightarrow C}{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2, \Phi_3 \Rightarrow \overrightarrow{B \bullet C}}\ \bullet R$$

$d(\mathcal{D}) < d(\mathcal{D}^*)$, for $w(B) + w(C) < w(B \bullet C)$.

- Suppose the last rule of the right premise is $\odot L$:

$$\mathcal{D} = \cfrac{\Delta \Rightarrow \overrightarrow{A} \quad \cfrac{\Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2, \overrightarrow{B}(\overrightarrow{C}(\Phi_3)), \Phi_4 \Rightarrow \overrightarrow{D}}{\Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2, \overrightarrow{B \odot C}(\Phi_3), \Phi_4 \Rightarrow \overrightarrow{D}}\ \odot L}{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2, \overrightarrow{B \odot C}(\Phi_3), \Phi_4 \Rightarrow \overrightarrow{D}}\ cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \cfrac{\cfrac{\Delta \Rightarrow A \quad \Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2, \overrightarrow{B}(\overrightarrow{C}(\Phi_3)), \Phi_4 \Rightarrow \overrightarrow{D}}{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2, \overrightarrow{B}(\overrightarrow{C}(\Phi_3)), \Phi_4 \Rightarrow \overrightarrow{D}}\ cut}{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2, \overrightarrow{B \odot C}(\Phi_3), \Phi_4 \Rightarrow \overrightarrow{D}}\ \odot L$$

$d(\mathcal{D}) < d(\mathcal{D}^*)$, for $w(B) + w(C) < w(B \odot C)$.

- Suppose the last rule of the right premise is $\odot R$:

$$\mathcal{D} = \frac{\Delta \Rightarrow A \quad \dfrac{\Phi_0, A(\Phi_1), \Phi_2 \Rightarrow B \quad \Phi_3 \Rightarrow C}{\overrightarrow{\Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2}(\Phi_3) \Rightarrow \overrightarrow{B \odot C}} \bullet R}{\overrightarrow{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2}(\Phi_3) \Rightarrow \overrightarrow{B \odot C}} cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \frac{\dfrac{\Delta \Rightarrow A \quad \Phi_0, \overrightarrow{A}(\Phi_1), \Phi_2 \Rightarrow B}{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2 \Rightarrow B} cut \qquad \Phi_3 \Rightarrow C}{\overrightarrow{\Phi_0, \overrightarrow{\Delta}(\Phi_1), \Phi_2}(\Phi_3) \Rightarrow \overrightarrow{B \odot C}} \odot R$$

$d(\mathcal{D}) < d(\mathcal{D}^*)$, for $w(B) + w(C) < w(B \odot C)$.

- Suppose the last rule of the right premise is $\breve{L}$:

$$\mathcal{D} = \frac{\Delta \Rightarrow A \quad \dfrac{\Phi_0, A(\Phi_1), \Phi_2, B, \Phi_3 \Rightarrow D}{\Phi_0, A(\Phi_1), \Phi_2, \sqrt[0]{\breve{B}}, \sqrt[1]{\breve{B}}, \Phi_3 \Rightarrow D} \breve{L}}{\Phi_0, \Delta(\Phi_1), \Phi_2, \sqrt[0]{\breve{B}}, \sqrt[1]{\breve{B}}, \Phi_3 \Rightarrow D} cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \frac{\dfrac{\Delta \Rightarrow A \quad \Phi_0, A(\Phi_1), \Phi_2, B, \Phi_3 \Rightarrow D}{\Phi_0, \Delta(\Phi_1), \Phi_2, B, \Phi_3 \Rightarrow D} cut}{\Phi_0, \Delta(\Phi_1), \Phi_2, \sqrt[0]{\breve{B}}, \sqrt[1]{\breve{B}}, \Phi_3 \Rightarrow D} \breve{L}$$

$d(\mathcal{D}) < d(\mathcal{D}^*)$, for $w(B) < w(\breve{B})$.

- Suppose the last rule of the right premise is $\breve{R}$:

$$\mathcal{D} = \frac{\Delta \Rightarrow A \quad \dfrac{\Phi_0, \Phi_1, A(\Phi_2), \Phi_3 \Rightarrow D}{\Phi_0, [\,], \Phi_1, A(\Phi_2), \Phi_3 \Rightarrow \breve{D}} \breve{R}}{\Phi_0, \Phi_1, \Delta(\Phi_2), \Phi_3 \Rightarrow \breve{D}} cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \cfrac{\cfrac{\Delta \Rightarrow A \quad \Phi_o, \Phi_1, A(\Phi_2), \Phi_3 \Rightarrow D}{\Phi_0, \Phi_1, \Delta(\Phi_2), \Phi_3}\,cut}{\Phi_0, [\,], \Phi_1, A(\Phi_2), \Phi_3 \Rightarrow D}\,\check{}R$$

$d(\mathcal{D}) < d(\mathcal{D}^*)$, *for* $w(B) < w(\check{}B)$.

- Suppose the last rule of the right premise is $\hat{}L$*:*

$$\mathcal{D} = \cfrac{\Delta \Rightarrow A \quad \cfrac{\Phi, \overrightarrow{A}(\Phi), \Phi, \sqrt[0]{B}, \sqrt[1]{B}, \Phi \Rightarrow D}{\Phi, \overrightarrow{A}(\Phi), \Phi, \hat{}B, \Phi \Rightarrow D}\,\hat{}L}{\Phi, \overrightarrow{\Delta}(\Phi), \Phi, \hat{}B, \Phi \Rightarrow D}\,cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \cfrac{\cfrac{\Delta \Rightarrow A \quad \Phi, \overrightarrow{A}(\Phi), \Phi, \sqrt[0]{B}, \sqrt[1]{B}, \Phi \Rightarrow D}{\Phi, \overrightarrow{A}(\Phi), \Phi, \sqrt[0]{B}, \sqrt[1]{B}, \Phi \Rightarrow D}\,cut}{\Phi, \overrightarrow{\Delta}(\Phi), \Phi, \hat{}B, \Phi \Rightarrow D}\,\hat{}L$$

$d(\mathcal{D}) < d(\mathcal{D}^*)$, *for* $w(B) < w(\hat{}B)$.

- Suppose the last rule of the right premise is $\hat{}R$*:*

$$\mathcal{D} = \cfrac{\Delta \Rightarrow A \quad \cfrac{\Phi_0, \Delta(\Phi_1), \Phi_2, [\,], \Phi_3 \Rightarrow B}{\Phi_0, A(\Phi_1), \Phi_2, \Phi_3 \Rightarrow \hat{}B}\,\hat{}R}{\Phi_0, \Delta(\Phi_1), \Phi_2, \Phi_3 \Rightarrow \hat{}B}\,cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \cfrac{\cfrac{\Delta \Rightarrow A \quad \Phi_0, A(\Phi_1), \Phi_2, [\,], \Phi_3 \Rightarrow B}{\Phi_0, \Delta(\Phi_1), \Phi_2, [\,], \Phi_3 \Rightarrow B}\,cut}{\Phi_0, \Delta(\Phi_1), \Phi_2, \Phi_3 \Rightarrow \hat{}B}\,\hat{}R$$

$d(\mathcal{D}) < d(\mathcal{D}^*)$, *for* $w(B) < w(\hat{}B)$.

- ***Principal cut case***:

  This means that the cut formula is the active formula in both premises. In this case, the transformation is of the derivation into another one with two cuts in which clearly their cut degrees are smaller, for some material is lost, namely the contribution of the connective building the cut formula.

  - Suppose that in the left premise the last rule is $\bullet R$ and in the right premise the last rule is $\bullet L$:

$$\mathcal{D} = \dfrac{\dfrac{\Delta_1 \Rightarrow \overrightarrow{A} \quad \Delta_2, [\,], \Delta_3 \Rightarrow \overrightarrow{B}}{\Delta_1, \Delta_2, [\,], \Delta_3 \Rightarrow \overrightarrow{A \bullet B}}\,\bullet R \quad \dfrac{\Phi_0, A, \overrightarrow{B}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}{\Phi_0, \overrightarrow{A \bullet B}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}\,\bullet L}{\Phi_1, \Delta_1, \Delta_2, \Phi_1, \Delta_3, \Phi_2 \Rightarrow \overrightarrow{D}}\,cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \dfrac{\Delta_2, [\,], \Delta_3 \Rightarrow \sqrt[0]{B}, [\,], \sqrt[1]{B} \quad \dfrac{\Delta_1 \Rightarrow A \quad \Phi_0, A, \overrightarrow{B}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}{\Phi_0, \Delta, \overrightarrow{B}(\Phi_1), \Phi_2 \Rightarrow \overrightarrow{D}}\,cut}{\Phi_1, \Delta_1, \Phi_1, \Delta_2, \Delta_3, \Phi_2 \Rightarrow \overrightarrow{D}}\,cut$$

  - Suppose in the left premise the last rule is $\backslash R$ and in the right premise the last rule is $\backslash L$:

$$\mathcal{D} = \dfrac{\dfrac{\overrightarrow{B}, \Delta \Rightarrow \overrightarrow{C}}{\Delta \Rightarrow \overrightarrow{B\backslash C}}\,\backslash R \quad \dfrac{\Phi_1 \Rightarrow B \quad \Phi_0, \overrightarrow{C}(\Phi_2), \Phi_3 \Rightarrow \overrightarrow{D}}{\Phi_0, \overrightarrow{\Phi_1, B\backslash C}(\Phi_2), \Phi_3 \Rightarrow D}\,\backslash L}{\Phi_0, \overrightarrow{\Phi_1, \Delta}(\Phi_2), \Phi_3 \Rightarrow D}\,cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \dfrac{\Phi_1 \Rightarrow \overrightarrow{B} \quad \dfrac{\overrightarrow{B}, \Delta \Rightarrow \overrightarrow{C} \quad \Phi_0, \overrightarrow{C}(\Phi_2), \Phi_3 \Rightarrow \overrightarrow{D}}{\Phi_0, \overrightarrow{\overrightarrow{B}, \Delta}(\Phi_2), \Phi_3 \Rightarrow \overrightarrow{D}}}{\Phi_0, \overrightarrow{\Phi_1, \Delta}(\Phi_2), \Phi_3 \Rightarrow \overrightarrow{D}}\,cut$$

The principal cut with / is completely similar to \.

- Suppose in the left premise the last rule is $\odot R$ *and in the right premise the last rule is* $\odot L$*:*

$$\mathcal{D} = \cfrac{\cfrac{\Delta_0,[\,],\Delta_2 \Rightarrow \overrightarrow{A}}{\Delta_0,\Delta_1,\Delta_2 \Rightarrow \overrightarrow{A\odot B}}\odot R \quad \cfrac{\Phi_0,\overrightarrow{A}(\overrightarrow{B}(\Phi_1)),\Phi_2 \Rightarrow \overrightarrow{D}}{\Phi_0,\overrightarrow{A\odot B}(\Phi_1),\Phi_2\Rightarrow\overrightarrow{D}}\odot L}{\Phi_0,\Delta_0,\overrightarrow{\Delta_1}(\Phi_1),\Delta_2,\Phi_2\Rightarrow\overrightarrow{D}}\, cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \cfrac{\Delta_0,[\,],\Delta_2 \Rightarrow \overrightarrow{A} \quad \cfrac{\Delta_1 \Rightarrow \overrightarrow{B} \quad \Phi_0,\overrightarrow{A}(\overrightarrow{B}(\Phi_1)),\Phi_2 \Rightarrow \overrightarrow{D}}{\Phi_0,\overrightarrow{A}(\overrightarrow{\Delta_1}(\Phi_1)),\Phi_2\Rightarrow\overrightarrow{D}}\, cut}{\Phi_0,\Delta_0,\overrightarrow{\Delta_1}(\Phi_1),\Delta_2,\Phi_2\Rightarrow\overrightarrow{D}}\, cut$$

- Suppose in the left premise the last rule is $\uparrow R$ *and in the right premise the last rule is* $\uparrow L$*:*

$$\mathcal{D} = \cfrac{\cfrac{\Delta_0,\overrightarrow{B},\Delta_2 \Rightarrow C}{\Delta_0,[\,],\Delta_2 \Rightarrow C\uparrow B}\uparrow R \quad \cfrac{\Gamma \Rightarrow B \quad \Phi_0,C(\Phi_1),\Phi_2 \Rightarrow D}{\Phi_0,\overrightarrow{C\uparrow B}(\overrightarrow{\Gamma}(\Phi_1)),\Phi_2\Rightarrow D}\uparrow L}{\Phi_0,\Delta_0,\overrightarrow{\Gamma}(\Phi_1),\Delta_2,\Phi_2\Rightarrow D}\, cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \cfrac{\Gamma \Rightarrow B \quad \cfrac{\Delta_0,\overrightarrow{B},\Delta_2 \Rightarrow C \quad \Phi_0,A(\Phi_1),\Phi_2 \Rightarrow D}{\Phi_0,\Delta_1,B,\Delta_2(\Phi_1),\Phi_2\Rightarrow D}\, cut}{\Phi_0,\overrightarrow{\Delta_0,\Gamma,\Delta_2}(\Phi_1),\Phi_2\Rightarrow D}\, cut$$

Now, the last sequent of the derivation above is identical to $\Phi_0,\Delta_0,\overrightarrow{\Gamma}(\Phi_1),\Delta_2,\Phi_2 \Rightarrow \overrightarrow{D}$

- *Suppose in the left premise the last rule is $\hat{}R$ and in the right premise the last rule is $\hat{}L$:*

$$\mathcal{D} = \dfrac{\dfrac{\Delta_0, [\,], \Delta_1 \Rightarrow \sqrt[0]{A}, [\,], \sqrt[1]{A}}{\Delta_0, \Delta_1 \Rightarrow \hat{}A}\,\hat{}R \quad \dfrac{\Phi_0, \sqrt[0]{A}, \sqrt[1]{A}, \Phi_2 \Rightarrow \overrightarrow{D}}{\Phi_0, \hat{}A, \Phi_2 \Rightarrow \overrightarrow{D}}\,\hat{}L}{\Phi_0, \Delta_0, \Delta_1, \Phi_1 \Rightarrow D}\,cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \dfrac{\Delta_0, [\,], \Delta_1 \Rightarrow \sqrt[0]{A}, [\,], \sqrt[1]{A} \quad \Phi_0, \sqrt[0]{A}, \sqrt[1]{A}, \Phi_2 \Rightarrow \overrightarrow{D}}{\Phi_0, \Delta_0, \Delta_1, \Phi_1 \Rightarrow D}\,cut$$

- *Suppose in the left premise the last rule is $\check{}R$ and in the right premise the last rule is $\check{}L$:*

$$\mathcal{D} = \dfrac{\dfrac{\Delta_0, \Delta_1 \Rightarrow A}{\Delta_0, [\,], \Delta_1 \Rightarrow \check{}A}\,\check{}R \quad \dfrac{\Phi_0, A, \Phi_2 \Rightarrow D}{\Phi_0, \sqrt[0]{\check{}A}, \sqrt[1]{\check{}A}, \Phi_2 \Rightarrow D}\,\check{}L}{\Phi_0, \Delta_0, \Delta_1, \Phi_1 \Rightarrow D}\,cut$$

$$\rightsquigarrow$$

$$\mathcal{D}^* = \dfrac{\Delta_0, \Delta_1 \Rightarrow A \quad \Phi_0, A, \Phi_2 \Rightarrow D}{\Phi_0, \Delta_0, \Delta_1, \Phi_1 \Rightarrow D}\,cut$$

Remark 10. *Cases with units I, J are completely straightforward.*

$\square$

2.3 Consequences

We give some nice consequences of the cut elimination theorem:

Theorem 2 (subformula property)**.** *Let $\mathcal{S} = \Delta \Rightarrow \overrightarrow{A}$ be a provable hypersequent of 1-**DLC** (without unit. Then, all the formulas building a cut-free derivation of $\mathcal{S}$ are subformulas of $\mathcal{S}$.*

Proof. All the rules of 1-**DLC** have this property. □

Theorem 3 (decidability for 1-**DLC**)**.** *Let $\mathcal{S} = \Delta \Rightarrow \overrightarrow{A}$ be a hypersequent of 1-**DLC** (without unit) with cut. The provability of $\mathcal{S}$ is decidable.*

Proof. Any proof derivation $\mathcal{D}$ with cut may be turned into one cut-free by the cut-elimination theorem. The subformula property of 1-**DLC** leads to a finite search space, whence provability is decidable. □

Theorem 4 (decidability for 1-**DLC**ϵ)**.** *Let $\mathcal{S} = \Delta \Rightarrow \overrightarrow{A}$ be a hypersequent of 1-**DLC**ϵ (with units) with cut. The provability of $\mathcal{S}$ is decidable.*

Proof. In some sense, the left rule of units preserve the subformula property, for they simply disappear. The search space in a cut-free proof in 1-**DLC**ϵ is still finite, whence provability is decidable. □

We turn back now to the categorical 1-discontinuous syntactic calculus of the last chapter. It is decidable:

Theorem 5 (decidability for the combinatory 1-discontinuous syntactic calculus)**.** *Let $A \Rightarrow B$ be a sequent of the combinatory 1-discontinuous syntactic calculus. The provability of $A \Rightarrow B$ is decidable.*

Proof. Translate $A \Rightarrow B$ via τ[2]. Then, by theorem 4, $\vdash \tau(A \Rightarrow B)$ is decidable. □

2.4 Another point

By simple application of the cut-formula, we deduce that hypersequents of the form $\Delta \Rightarrow B|A$, where $|$ is any implicative connective, are invertible. For example, if $\Delta \Rightarrow B \uparrow A$, we have $\overrightarrow{B \uparrow A}(A) \Rightarrow \overrightarrow{B}$, and thus by cut, $\overrightarrow{\Delta}(A) \Rightarrow \overrightarrow{B}$. This hypersequent has moreover a cut-free proof derivation (by the cut-elimination theorem).

[2]See the last chapter.

2.5 Conclusions

*We have proved a fundamental theorem: the cut-elimination theorem for the 1-**DLC** and the 1-**DLC**ϵ as well as for the 1-**DLC**$\hat{}, \check{}$. For the proof, a new way of measuring the weight of configurations has been provided. This theorem has interesting consequences for the decidability of 1-discontinuous categorical Lambek Calculus, and gives the invertibility of some logical rules.*

Chapter 3

Prosodic interpretation

In this chapter, we present the way to interpret sorted types as well as hypersequents[1]*. This gives consistence to our model-theoretic approach to discontinuity. The algebraic sorted operations in the power-set algebra of a 1-graded monoid are used to interpret types and the antecedents of hypersequents. Soundness is proved and several completeness results are proved. Finally, a source of incompleteness is shown. It is interesting to remark the construction of a canonical model for the implicative (continuous and discontinuous) fragment.*

[1] which are sorted as well.

3.1 Prosodic interpretation of the hypersequent calculus in 1-graded monoids

Type-logical linguists are mainly interested in the prosodic interpretations of types in familiar structures like semigroups or monoids (for the continuous case) and we claim that 1-graded monoids are good candidates for dealing with discontinuity phenomena. In chapter 1, we saw a 1-graded monoid in which types are interpreted as sorted subsets of the power set of $\widetilde{V}$, $\mathcal{P}(\widetilde{V})$. *We will interpret hypersequents in* $\mathcal{P}(\widetilde{V})$.

A continuous power-set frame over a monoid $\langle V, \cdot, I\rangle$, *is the power-set residuated monoid* $\langle \mathcal{P}(V), \circ, \backslash\backslash, //, \mathbb{I}; \subseteq\rangle$. *A discontinuous power-set frame over a monoid is the power-set residuated 1-graded monoid* $\langle \mathcal{P}(V_0 \cup (V_0 \cdot \$ \cdot V_0)), \circ, \backslash\backslash, //, \hat{\circ}, \Uparrow, \Downarrow, \mathbb{I}, \mathbb{J}; \subseteq\rangle$.

A continuous power-set model is a power-set frame $(\langle \mathcal{P}(V), \circ, \backslash\backslash\ ; \subseteq\rangle, [\![\cdot]\!]) = (\mathbb{F}_{cont}, [\![\cdot]\!])$ *with a valuation on types defined recursively over atomic variables. Similarly, a discontinuous power-set model is a discontinuous power-set frame* $(\langle \mathcal{P}(V_0 \cup (V_0 \cdot \$ \cdot V_0)), \circ, \backslash\backslash, //, \hat{\circ}, \Uparrow, \Downarrow, \mathbb{I}, \mathbb{J}; \subseteq\rangle, [\![\cdot]\!]) = (\mathbb{F}_{discont}, [\![\cdot]\!])$ *with a valuation on the set of (continuous and discontinuous) types defined recursively over atomic variables of sort 0 or 1.*

In the following definition, we use the algebraic operations on subsets of 1-graded monoids defined in chapter 1. $\widetilde{V} = V_0 \cup V_1$ *denotes a 1-graded monoid over a monoid* V_0. $\$$ *denotes its separator.*

Definition 12 (type interpretation).

$[\![A]\!] \subseteq V_0$ *if* A *atomic of sort 0*

$[\![A]\!] \subseteq V_1$ *if* A *atomic of sort 1*

$[\![A \bullet B]\!] = [\![A]\!] \circ [\![B]\!]$

$[\![B/A]\!] = [\![B]\!]//[\![A]\!]$

$[\![A\backslash B]\!] = [\![A]\!]\backslash\backslash[\![B]\!]$

$[\![A \odot B]\!] = [\![A]\!] \hat{\circ} [\![B]\!]$

$[\![B \uparrow A]\!] = [\![B]\!] \Uparrow [\![A]\!]$

$[\![A \downarrow B]\!] = [\![A]\!] \Downarrow [\![B]\!]$

$[\![I]\!] = \mathbb{I}$

$[\![J]\!] = \mathbb{J}$

Definition 13 (prosodic interpretation of an antecedent configuration)**.** *Let* Δ *be an antecedent configuration. Given a discontinuous model* $(\mathbb{F}, [\![\cdot]\!])$*, we define* $[\![\Delta]\!]_{\mathbb{F}}$ *recursively on the structure of an antecedent configuration in the following way:*

If $\Delta = \Lambda$*:*

$$[\![\Delta]\!]_{\mathbb{F}} = [\![I]\!]_{\mathbb{F}}$$

If $\Delta = E$ *for a type* E *of sort 0:*

$$[\![\Delta]\!]_{\mathbb{F}} = [\![E]\!]_{\mathbb{F}}$$

If $\Delta = \Gamma_1, \Gamma_2$*, with* Γ_i *of sort 0:*

$$[\![\Delta]\!]_{\mathbb{F}} = [\![\Gamma_1]\!]_{\mathbb{F}} \circ [\![\Gamma_2]\!]_{\mathbb{F}}$$

If $\Delta = \Gamma_0, \sqrt[0]{A}, \Gamma_1, \sqrt[1]{A}, \Gamma_2$*, with* A *an arbitrary type of sort 1, and* Γ_i *of sort 0:*

$$[\![\Delta]\!]_{\mathbb{F}} = [\![\Gamma_0]\!]_{\mathbb{F}} \circ ([\![A]\!]_{\mathbb{F}} \hat{\circ} [\![\Gamma_1]\!]_{\mathbb{F}}) \circ [\![\Gamma_2]\!]_{\mathbb{F}}$$

If $\Delta = \Gamma_1, [\,], \Gamma_2$ *(Sort 1) with* Γ_i *of sort 0:*

$$[\![[\,]]\!]_{\mathbb{F}} = [\![J]\!]_{\mathbb{F}}$$

$$[\![\Delta]\!]_{\mathbb{F}} = [\![\Gamma_1]\!]_{\mathbb{F}} \circ [\![[\,]]\!]_{\mathbb{F}} \circ [\![\Gamma_2]\!]_{\mathbb{F}}$$

If $\Delta = \Gamma_0, \sqrt[0]{A}, \Gamma_1, \sqrt[1]{A}, \Gamma_2$*, with* A*,* Γ_1 *of sort 1, and* Γ_0 *and* Γ_2 *of sort 0:*

$$[\![\Delta]\!]_{\mathbb{F}} = [\![\Gamma_0]\!]_{\mathbb{F}} \circ ([\![A]\!]_{\mathbb{F}} \,\hat{\circ}\, [\![\Gamma_1]\!]_{\mathbb{F}}) \circ [\![\Gamma_2]\!]_{\mathbb{F}}$$

Other cases for sort 1 configurations are similar.

Remark 11. *In particular, if* E *is a type of sort 1,* $[\![\sqrt[0]{E}, [\,], \sqrt[1]{E}]\!]_{\mathbb{F}} = [\![E]\!]_{\mathbb{F}}$

Definition 14 (interpretation of hypersequents)**.** *Let* $\mathcal{S} = \Delta \Rightarrow \overrightarrow{A}$ *be a hypersequent. Given a discontinuous model* $(\mathbb{F}, [\![\cdot]\!])$*,* $[\![\mathcal{S}]\!]_{\mathbb{F}}$ *is defined as:*

$$[\![\mathcal{S}]\!]_{\mathbb{F}} = [\![\Delta]\!]_{\mathbb{F}} \cap [\![A]\!]_{\mathbb{F}}$$

Definition 15 (truth w.r.t an L-model)**.** *Let* $\mathcal{S} = \Delta \Rightarrow \overrightarrow{A}$ *be a hypersequent. Given a discontinuous model* $(\mathbb{F}, [\![\cdot]\!])$*:*

$$\mathbb{F} \models \mathcal{S} \text{ iff } [\![\mathcal{S}]\!]_{\mathbb{F}} = [\![\Delta]\!]_{\mathbb{F}}$$

Remark 12. *The definition 15 is equivalent to* $[\![\Delta]\!]_{\mathbb{F}} \subseteq [\![A]\!]_{\mathbb{F}}$*.*

Definition 16 (validity)**.** *Let* $\mathcal{S} = \Delta \Rightarrow A$ *be a hypersequent:*

$$\models \mathcal{S} \text{ iff } \forall(\mathbb{F}, [\![\cdot]\!]) \;\; \mathbb{F} \models \mathcal{S}$$

Given a discontinuous power-set model $(\mathbb{F}, [\![\cdot]\!])$*, we will usually write* $[\![\cdot]\!]$ *instead of* $[\![\cdot]\!]_{\mathbb{F}}$*, if the frame* $\mathbb{F}$ *is clear from the context.*

Theorem 6 (soundness w.r.t power-set residuated 1-graded monoids)**.** *Let* $\mathcal{S} = \Delta \Rightarrow \overrightarrow{A}$ *be a hypersequent. Then:*

$$\models \mathcal{S} \text{ if } \vdash \mathcal{S}$$

Proof. By induction on the length of a hypersequent derivation. □

3.2 Some results on completeness and incompleteness

In this section, we prove two completeness results. The first one refers to the interpretation in free 1-graded monoids (see definition 17) of the implicative fragment (continuous and discontinuous connectives) without units. The fragment with both units is shown to be incomplete w.r.t the class of free 1-graded monoids. The second completeness result is for the full fragment, but this time, the interpretation is in the class of preordered 1-graded monoids.

Definition 17. *Let A be a free monoid. Let \$ be an element of A different from the unit of A. V_0 denotes the free submonoid generated by $A - \{\$\}$. Then, $\widetilde{V} = V_0 \cup V_1$, where $V_1 = V_0 \cdot \$ \cdot V_0$ is called the free 1-graded monoid of A with separator \$.*

Definition 18 (discontinuous L-models)**.** *Let A be a free monoid with separator \$ like in the previous definition. The residuated power-set 1-graded monoid over $\widetilde{V}$ with a valuation $[\![\cdot]\!]$, $(\mathbb{F}, [\![\cdot]\!])$ is called a discontinuous language model or (discontinuous) L-model.*

3.2.1 A completeness result for discontinuous L-models

Let $T ::= \mathcal{F}_0 \,|\, \sqrt[0]{\mathcal{F}_1} \,|\, \sqrt[1]{\mathcal{F}_1}$ where $\sqrt[0]{}\ \sqrt[1]{}$ are the symbols used for the components of a sort 1 type. The free 1-graded monoid over $T \cup \{[\,]\}$ with separator[2] $[\,]$, is denoted $\widetilde{T}$. As usual, $\widetilde{V} = T^* \cup (T^* \cdot [\,] \cdot T^*)$ (T^* is the Kleene closure over T). Let $\mathbb{T}$ be the discontinuous power-set frame over $\widetilde{T}$.

Remark 13. *It's important to observe that the set of (correct) configurations $\mathcal{O} \subsetneq \widetilde{T}$.*

Definition 19. *Let $(\mathbb{T}, [\![\cdot]\!])$ be the discontinuous L-model over the free 1-graded monoid $\widetilde{T}$ defined above. Let us define $[\![\cdot]\!]$ on (sort 0 or sort 1) atomic types by:*

$$\textit{For every atomic } A \in \mathcal{A},\ [\![A]\!] = \{\Delta :\ \Delta \in \widetilde{V} \ \&\ \vdash \Delta \Rightarrow \overrightarrow{A}\}$$

[2] As we know, $[\,] \notin T^*$

$\mathbb{CIM} = (\mathbb{T}, [\![\cdot]\!])$ *is called the **canonical implicative model**.*

Lemma 4 (identity)**.** *For every $A \in \mathcal{F}$,*

$$\vdash A \Rightarrow A, \text{ if } A \text{ is of sort } 0$$

$$\vdash \sqrt[0]{A}, [\,], \sqrt[1]{A} \Rightarrow \sqrt[0]{A}, [\,], \sqrt[1]{A}, \text{ if } A \text{ of sort } 1$$

Proof. By induction on the complexity of A. □

Lemma 5 (truth Lemma)**.** *Let $\mathbb{CIM} =< \mathbb{T}, [\![\cdot]\!] >$ be the canonical implicative model defined above. Then:*

a) For every type A (of any sort), $[\![A]\!] = \{\Delta : \Delta \in \widetilde{T} \ \& \ \vdash \Delta \Rightarrow \overrightarrow{A}\}$.

b) For every $\Delta \in \mathcal{O}$, $\Delta \in [\![\Delta]\!]$.

Proof. a) In order to have a less painful proof, we restrict it to the sort functionalities $\backslash_{(0,0)\to 0}, /_{(0,0)\to 0}, \uparrow_{(0,0)\to 1}, \downarrow_{(1,0)\to 0}$. Cases with other sort functionalities, mimic the proof of the restricted case.

Continuous connectives follow the standard proof [Bus82][Buszkowski 86] in (continuous) Lambek calculus. Let's see the proof for the discontinuous connectives.

Discontinuous connectives:

1. "Extract" connective: For A and B types of sort 0, we have to see that $[\![B \uparrow A]\!] = \{(\Delta, [\,], \Gamma) : \vdash \Delta, [\,], \Gamma \Rightarrow B \uparrow A\}$, that is,

$$[\![B]\!] \Uparrow [\![A]\!] = \{(\Delta, [\,], \Gamma) : \vdash \Delta, [\,], \Gamma \Rightarrow B \uparrow A\}$$

Let us see $\subseteq$:

Let $(\Delta, [\,], \Gamma) \in [\![B \uparrow A]\!] = [\![B]\!] \Uparrow [\![A]\!]$. This means that for every $\Phi \in [\![A]\!]$, $\Delta, \Phi, \Gamma \in [\![B]\!]$, and then by induction hypothesis (i.h), $\vdash \Delta, \Phi, \Gamma \Rightarrow B$. If $\Phi = A$, then $\vdash \Delta, A, \Gamma \Rightarrow B$, and by application of the right rule for $\uparrow$, $\vdash \Delta, [\,], \Gamma \Rightarrow B \uparrow A$.

For $\supseteq$, consider $(\Delta, [\,], \Gamma)$ such that $\vdash \Delta, [\,], \Gamma \Rightarrow B \uparrow A$. We have to see that $(\Delta, [\,], \Gamma) \in [\![B]\!] \Uparrow [\![A]\!]$, that is, for every $\Omega \in [\![A]\!]$, $\Delta, \Omega, \Gamma \in [\![B]\!]$. By i.h, $\vdash \Omega \Rightarrow A$. Moreover, $\vdash \sqrt[0]{B \uparrow A}, A, \sqrt[1]{B \uparrow A} \Rightarrow B$. Thus, by the *cut* rule, $\vdash \Delta, \Omega, \Gamma \Rightarrow B$, and so, by i.h., $\Delta, \Omega, \Gamma \in [\![B]\!]$.

2. "Infix" connective: $[\![A \downarrow B]\!] = \{\Delta : \vdash \Delta \Rightarrow A \downarrow B\}$, that is,

$$[\![A]\!] \Downarrow [\![B]\!] = \{\Delta : \vdash \Delta \Rightarrow A \downarrow B\}$$

Let us see $\subseteq$:

Let $\Delta \in [\![A]\!] \Downarrow [\![B]\!]$. Then, for every $(\Gamma, [\,], \Omega) \in [\![A]\!]$, $\Gamma, \Delta, \Omega \in [\![B]\!]$. By i.h, $\vdash \Gamma, [\,], \Omega \Rightarrow A$. In particular, by lemma 4, $\vdash \sqrt[0]{A}, [\,], \sqrt[1]{A} \Rightarrow \overrightarrow{A}$, and then, $\vdash \sqrt[0]{A}, \Delta, \sqrt[1]{A} \Rightarrow B$. By the right rule of $\downarrow$, $\vdash \Delta \Rightarrow A \downarrow B$.

For $\supseteq$, consider Δ such that $\vdash \Delta \Rightarrow A \downarrow B$. We have to see that $\Delta \in [\![A]\!] \Downarrow [\![B]\!]$, which means that for every $(\Gamma, [\,], \Omega) \in [\![A]\!]$, $\Gamma, \Delta, \Omega \in [\![B]\!]$. By induction hypothesis, $\vdash \Gamma, [\,], \Omega \Rightarrow \overrightarrow{A}$. By simple application, $\sqrt[0]{A}, A \downarrow B, \sqrt[1]{A} \Rightarrow B$. Then, by cut with premises $\Gamma, [\,], \Omega \Rightarrow \overrightarrow{A}$ and $\sqrt[0]{A}, A \downarrow B, \sqrt[1]{A} \Rightarrow B$, $\vdash \Gamma, A \downarrow B, \Omega \Rightarrow B$. Again by cut with premises $\Gamma, A \downarrow B, \Omega \Rightarrow B$ and $\Delta \Rightarrow A \Downarrow B$, $\vdash, \Gamma, \Delta, \Omega \Rightarrow B$. By i.h, $\Gamma, \Delta, \Omega \in [\![B]\!]$.

b) Proof by induction on the complexity of configurations:

- If $\Delta = \Lambda$, $\Lambda \in [\![\Lambda]\!]$.

- If $\Delta = E$, where E is of sort 0:

$\vdash E \Rightarrow E$, so by a), $E \in [\![E]\!] = [\![\Delta]\!]$.

- If $\Delta = \Gamma_1, \Gamma_2$, with Γ_i of sort 0:

By i.h, $\Gamma_i \in [\![\Gamma_i]\!]$. So, $\Gamma_1, \Gamma_2 = \Gamma_1 \cdot \Gamma_2 \in [\![\Gamma_1]\!] \circ [\![\Gamma_2]\!] = [\![\Gamma_1, \Gamma_2]\!]$.

- If $\Delta = \Gamma_1, \sqrt[0]{A}, \Gamma_2, \sqrt[1]{A}, \Gamma_3$, with A an arbitrary type of sort 1, and Γ_i of sort 0:

$$[\![\Delta]\!] = [\![\Gamma_1]\!] \circ ([\![A]\!] \,\hat{\circ}\, [\![\Gamma_2)]\!] \circ [\![\Gamma_3]\!]$$

By lemma 4, $[\![\sqrt[0]{A}, [\,], \sqrt[1]{A}]\!] \in [\![A]\!]$. By i.h, $\Gamma_i \in [\![\Gamma_i]\!]$. Then,
$\Gamma_1 \cdot ((\sqrt[0]{A} \cdot [\,] \cdot \sqrt[1]{A}) \,\hat{\cdot}\, \Gamma_2) \cdot \Gamma_3 = \Gamma_1 \cdot \sqrt[0]{A} \cdot \Gamma_2 \cdot \sqrt[1]{A} \cdot \Gamma_3 =$
$\Gamma_1, \sqrt[0]{A}, \Gamma_2, \sqrt[1]{A}, \Gamma_3 \in [\![\Delta]\!] = [\![\Gamma_1]\!] \circ ([\![A]\!] \,\hat{\circ}\, [\![\Gamma_2)]\!] \circ [\![\Gamma_3]\!] = [\![\Delta]\!]$.

- If $\Delta = \Gamma_1, [\,], \Gamma_2$, with Γ_i of sort 0:

$$[\![\Delta]\!] = [\![\Gamma_1]\!] \circ [\![[\,]]\!] \circ [\![\Gamma_2]\!]$$

By i.h, $\Gamma_i \in [\![\Gamma_i]\!]$. $[\,] \in [\![[\,]]\!]$, so $\Gamma_1 \cdot [\,] \cdot \Gamma_2 \in [\![\Gamma_1]\!] \circ [\![[\,]]\!] \circ [\![\Gamma_2]\!] = [\![\Delta]\!]$.

- Finally, if $\Delta = \Gamma_1, \sqrt[0]{A}, \Theta, \sqrt[1]{A}, \Gamma_2$, with A, Θ of sort 1, and Γ_i of sort 0:

$$[\![\Delta]\!] = [\![\Gamma_1]\!] \circ ([\![A]\!] \,\hat{\circ}\, [\![\Theta]\!]) \circ [\![\Gamma_2]\!]$$

By i.h, $\Gamma_i \in [\![\Gamma_i]\!]$, $\Theta \in [\![\Theta]\!]$. Moreover, $\sqrt[0]{A}, [\,], \sqrt[1]{A} \in [\![A]\!]$.
So, $\Gamma_1 \cdot ((\sqrt[0]{A} \cdot [\,] \cdot \sqrt[1]{A}) \,\hat{\cdot}\, \Theta) \cdot \Gamma_2 = \Gamma_1, \sqrt[0]{A}, \Theta, \sqrt[1]{A}, \Gamma_2 \in [\![\Gamma_1]\!] \circ ([\![A]\!] \,\hat{\circ}\, [\![\Theta]\!]) \circ [\![\Gamma_2]\!] =$
$[\![\Delta]\!]$.

For every case, we have proved that $\Delta \in [\![\Delta]\!]$. □

Theorem 7 (completeness of of the implicative fragment without units w.r.t L–models)**.** *Let $\mathcal{S} = \Delta \Rightarrow \overrightarrow{A}$ be a hypersequent. Then:*

$$\vdash \mathcal{S} \text{ if } \models \mathcal{S}$$

Proof. Let us suppose $\models \mathcal{S}$. In particular, $\mathbb{CIM} \models S$. Then, $\Delta \in [\![\Delta]\!]_{\mathbb{CIM}}$ by the **Truth Lemma** b). By soundness, $\Delta \in [\![A]\!]_{\mathbb{CIM}}$, and so, by **Truth lemma** a), $\vdash \Delta \Rightarrow \overrightarrow{A}$. □

*Does the truth lemma 5 work for the full fragment of 1-**DLC**? See next lemma:*

Lemma 6. *The truth lemma stated before fails for the full fragment $DL_{\{\bullet, \backslash, /, \odot, \uparrow, \downarrow\}}$ of the canonical model.*

Proof. Consider the continuous product $\bullet$. The truth lemma for $\bullet$ would be:

$$[\![B \bullet C]\!] = [\![B]\!] \circ [\![C]\!] = \{\Delta \cdot \Gamma :\ \Delta \Rightarrow B \text{ and } \Gamma \Rightarrow C\}$$

Consider $A, A\backslash(B \bullet C) \Rightarrow B \bullet\ C$. It is true that $\vdash A, A\backslash B \bullet C \Rightarrow B \bullet\ C$. Then, $A, A\backslash(B \bullet C) \in [\![B]\!] \circ [\![C]\!]$. So, $A, A\backslash(B \bullet C) = \Delta \cdot \Gamma$ such that $\Delta \in [\![B]\!]$ and $\Gamma \in [\![C]\!]$. Clearly this is not possible. So the truth lemma fails. □

*The proof for the implicative fragment of the 1-**DLC** with units 1-**DLC**ϵ doesn't work.*

Remark 14 (incompleteness of the implicative fragment with units w.r.t L-models)**.** *Consider $I \uparrow I$. The equality $[\![J]\!] = [\![I \uparrow I]\!]$ holds in the class of L-models. Thus, $\models J \Leftrightarrow I \uparrow I$ is true whereas $\nvdash_{1-\boldsymbol{DLC}\epsilon} I \uparrow I \Rightarrow J$. The corresponding hypersequent $\sqrt[0]{I \uparrow I}, [\,], \sqrt[1]{I \uparrow I} \Rightarrow \sqrt[0]{J}, [\,], \sqrt[1]{J}$ is underivable.*

Remark 15. *The last example for incompleteness no longer holds of the larger class of 1-graded monoids. For, there are models where $(a \cdot \$ \cdot b) \hat{\div} I = a \cdot b = I$ and both a and b are different from the unit*[3]

3.2.2 The multisuccedent discontinuous Lambek Calculus

In this section, following Pentus [Pen93a], we formulate a multisuccedent hypersequent calculus[4]*. Figure 3.1 shows the calculus. Moving to the multisuccedent calculus **1-DLC**$^{\mu}$ gives a way of skipping the inherent asymmetry of the single conclusion calculus **1-DLC** or **1-DLC**ϵ. This result corresponds to the multisuccedent versions of intuitionistic logic. As an interesting application to completeness results, we will build in the next section a canonical model based on the multisuccedent hypersequent calculus. This canonical model will give full completeness w.r.t a larger (than discontinuous L-models) class of discontinuous models (see 3.2.3).*

[3]Consider nilpotent elements. If a is nilpotent then $(a \cdot \$ \cdot a) \hat{\div} I = a \cdot a = I$ and $a \neq I$. Thus, in this model $[\![I \uparrow I]\!] \neq \{\$\}$.

[4] A remark on notation. In the following, $\Delta([\,])$ means a (sort 1) configuration (the separator $[\,]$ helps the reader to realize that the configuration is of sort 1).

$$\frac{}{\Theta \Rightarrow_\mu \Theta}\ id_0$$

$$\frac{}{\Theta([\,]) \Rightarrow_\mu \Theta([\,])}\ id_1$$

$$\frac{\Delta_0, \Delta_1 \Rightarrow_\mu \Theta}{\Delta_0, I, \Delta_1 \Rightarrow_\mu \Theta}\ IL \qquad \frac{\Delta \Rightarrow_\mu \Theta_0, \Theta_1}{\Delta \Rightarrow_\mu \Theta_0, I, \Theta_1}\ IR$$

$$\frac{\Delta_0, \Delta_1, \Delta_2 \Rightarrow_\mu \Theta}{\Delta_0, \sqrt[0]{J}, \Delta_1, \sqrt[1]{J}, \Delta_2 \Rightarrow_\mu \Theta}\ JL \qquad \frac{\Delta \Rightarrow_\mu \Theta_0, \Theta_1, \Theta_2}{\Delta \Rightarrow_\mu \Theta_0, \sqrt[0]{J}, \Theta_1, \sqrt[1]{J}, \Theta_2}\ JR$$

$$\frac{\Delta_1,\ A,\ B, \Delta_2 \Rightarrow_\mu \Sigma}{\Delta_1,\ A \bullet B,\ \Delta_2 \Rightarrow_\mu \Sigma}\ \bullet L \qquad \frac{\Delta \Rightarrow_\mu \Sigma_1, A, B, \Sigma_2}{\Delta \Rightarrow_\mu \Sigma_1, A \bullet B, \Sigma_2}\ \bullet R$$

$$\frac{\Gamma \Rightarrow_\mu A \quad \Delta_1,\ C,\ \Delta_2 \Rightarrow_\mu \Sigma}{\Delta_1, \Gamma,\ A\backslash C,\ \Delta_2 \Rightarrow_\mu \Sigma}\ \backslash L \qquad \frac{A,\ \Gamma \Rightarrow_\mu C}{\Gamma \Rightarrow_\mu A\backslash C}\ \backslash R$$

$$\frac{\Gamma \Rightarrow_\mu A \quad \Delta_1,\ C,\ \Delta_2 \Rightarrow_\mu \Sigma}{\Delta_1,\ C/A, \Gamma, \Delta_2 \Rightarrow_\mu \Sigma}\ /L \qquad \frac{\Gamma,\ A \Rightarrow_\mu C}{\Gamma \Rightarrow_\mu C/A}\ /R$$

$$\frac{\Delta_1,\ \sqrt[0]{A}^{(1)},\ B,\ \sqrt[1]{A}^{(1)}, \Delta_2 \Rightarrow_\mu \Sigma}{\Delta_1,\ A^{(1)} \odot B,\ \Delta_2 \Rightarrow_\mu \Sigma}\ \odot L \qquad \frac{\Delta \Rightarrow_\mu \Gamma_1,\ \sqrt[0]{A}^{(1)}, B,\ \sqrt[1]{A}^{(1)}, \Gamma_2}{\Delta \Rightarrow_\mu \Gamma_1, A \odot B, \Gamma_2}\ \odot R$$

$$\frac{\Gamma_1, [\,], \Gamma_2 \Rightarrow_\mu A \quad \Delta_1,\ C,\ \Delta_2 \Rightarrow_\mu \Sigma}{\Delta_1, \Gamma_1,\ A \downarrow B, \Gamma_2, \Delta_2 \Rightarrow_\mu \Sigma}\ \downarrow L \qquad \frac{\sqrt[0]{A},\ \Gamma,\ \sqrt[1]{A} \Rightarrow_\mu C}{\Gamma \Rightarrow_\mu A \downarrow C}\ \downarrow R$$

$$\frac{\Gamma \Rightarrow_\mu B \quad \Delta_1,\ C,\ \Delta_2 \Rightarrow_\mu \Sigma}{\Delta_1,\ \sqrt[0]{C \uparrow B}, \Gamma,\ \sqrt[1]{C \uparrow B} \Rightarrow_\mu \Sigma}\ \uparrow L \qquad \frac{\Gamma_1, B, \Gamma_2 \Rightarrow_\mu C}{\Gamma_1, [\,], \Gamma_2 \Rightarrow_\mu\ \sqrt[0]{(C \uparrow B)}, [\,],\ \sqrt[1]{(C \uparrow B)}}\ \uparrow R$$

$$\frac{\Gamma \Rightarrow_\mu \Theta \quad \Delta \Rightarrow_\mu \Phi}{\Gamma, \Delta \Rightarrow_\mu \Theta, \Phi}\ Con \qquad \frac{\Gamma([\,]) \Rightarrow_\mu \Theta([\,]) \quad \Delta \Rightarrow_\mu \Phi}{\Gamma(\Delta) \Rightarrow_\mu \Theta(\Phi)}\ Wrap$$

Figure 3.1: The 1-discontinuous multisuccedent Lambek Calculus

Remark 16. *The identity axioms includes the case* $[\,] \Rightarrow_\mu [\,]$*.*

Definition 20 (type equivalent of sort 0)**.** *Let* Δ *be a configuration of sort 0. We define* $\Delta^\bullet$ *as the process of eliminating occurrences of components* $\sqrt[0]{}, \sqrt[1]{}$ *by successive application of the left rule of the discontinuous connective product* $\odot$*.*

Example 2. *If* $\Delta = \sqrt[0]{A}, B, \sqrt[1]{A}$*, then* $\Delta^\bullet = A \odot B$*.*

Remark 17. *If* $\Delta = \sqrt[0]{A}, \sqrt[1]{A}$*, then* $\Delta^\bullet = A \odot I$*. We could use also use the unary bridge operator* $\hat{}$ *:* $\Delta^\bullet = \hat{}A$*.*

Lemma 7. *If* $\Delta \Rightarrow A$ *is a hypersequent of sort 0, then:*

$$\vdash \Delta \Rightarrow A \text{ iff } \Delta^\bullet \Rightarrow A$$

Proof. - Only if: $\Delta \Rightarrow A$ is a provable hypersequent, the we only have to apply successively the left rule of $\bullet$ and $\odot$.

- If:

It's obvious that $\Delta \Rightarrow \Delta^\bullet$ (Straightforward Induction on the sort 0 configuration). If $\Delta^\bullet \Rightarrow A$, then by cut, $\Delta \Rightarrow A$.

□

*What is the connection between the multisuccedent calculus 1-**DLC**μ and the single succedent calculus 1-**DLC**?*

Lemma 8. *For every* Δ, Φ*:*

- $\vdash_\mu \Delta \Rightarrow_\mu \Phi$ *iff* $\vdash \Delta \Rightarrow (\Phi)^\bullet$*.*
- $\vdash_\mu \Delta([\,]) \Rightarrow_\mu \Phi([\,])$ *iff* $\vdash (\Delta(C))^\bullet \Rightarrow_\mu (\Phi(C))^\bullet$*, for every* $C \in \mathcal{F}_0$*.*
- $\vdash_\mu \Delta([\,]) \Rightarrow_\mu \sqrt[0]{A}, [\,], \sqrt[1]{A}$ *iff* $\vdash \Delta([\,]) \Rightarrow A$

Proof.

- *only if* case: Induction on the length of a DLC^μ derivation.

1. Axioms:

$$\Delta \Rightarrow_{\mu} \Delta \text{ and } \Delta([\,]) \Rightarrow_{\mu} \Delta([\,])$$

For every $\Delta \in \mathcal{O}_0$, $\Delta \Rightarrow (\Delta)^{\bullet}$. Similarly, for every $\Delta([\,]) \in \mathcal{O}_1$, and $C \in \mathcal{F}$ $\Delta(C) \Rightarrow (\Delta(C))^{\bullet}$. The proof is by induction on the complexity of a configuration. Moreover, if Δ or $\Delta([\,])$ are equal respectively to A or $\overrightarrow{A}$, the claims hold, namely, $\overrightarrow{A} \Rightarrow_{\mu} \overrightarrow{A}$, then $\overrightarrow{A} \Rightarrow \overrightarrow{A}$

2. Right rules. Consider $\downarrow$. By i.h, $\overrightarrow{A}(\Delta) \Rightarrow B$, so $\Delta \Rightarrow A \downarrow B$. Left rules: By i.h, $\Gamma \Rightarrow A$ and $\Delta(B) \Rightarrow (\Theta)^{\bullet}$, thus by the left rule for $\Rightarrow$, $\Delta(\Gamma(A \downarrow B))(\Theta)^{\bullet}$.

   Consider continuous product. The left rule case is obvious. Now, the right rule case. By, i.h:

$$\Delta \Rightarrow (\Theta(A, B))^{\bullet}$$

   We want to prove $\Delta \Rightarrow (\Theta(A \bullet B))^{\bullet}$. In this case, we apply cut with $(\Theta(A, B))^{\bullet} \Rightarrow (\Theta(A \bullet B))^{\bullet}$. For the discontinuous product, by i.h:

$$\Delta \Rightarrow (\Theta(\overrightarrow{A}(B)))^{\bullet}$$

   Again, we apply cut with $(\Theta(\overrightarrow{A}(B)))^{\bullet} \Rightarrow (\Theta(A \odot B))^{\bullet}$.

3. **Con** rule:

   By i.h, $\Delta \Rightarrow \Gamma^{\bullet}$ and $\Theta \Rightarrow \Sigma^{\bullet}$. By $\bullet R$, $\Delta, \Gamma \Rightarrow (\Theta^{\bullet}, \Sigma^{\bullet})^{\bullet}$. By application of cut with $(\Theta^{\bullet}, \Sigma^{\bullet})^{\bullet} \Rightarrow (\Theta, \Sigma)^{\bullet}$.

   Wrap Rule:

   By i.h, for every $C \in \mathcal{F}$, $\Delta(C) \Rightarrow (\Gamma(C))^{\bullet}$ and $\Theta \Rightarrow (\Sigma)^{\bullet}$. In particular, $\Delta((\Theta)^{\bullet}) \Rightarrow (\Gamma(\Theta^{\bullet}))^{\bullet}$, so $(\Delta((\Theta)^{\bullet}))^{\bullet} \Rightarrow (\Gamma(\Theta^{\bullet}))^{\bullet}$. $\Delta(\Sigma) \Rightarrow (\Delta((\Theta)^{\bullet}))^{\bullet}$. By (Cut), $\Delta(\Sigma) \Rightarrow (\Gamma(\Theta)^{\bullet})$. Finally, $(\Gamma((\Theta^{\bullet}))^{\bullet} \Rightarrow (\Gamma(\Theta))^{\bullet}$.

- *if* case: Straightforward induction on the length of a derivation of a sequent of the single succedent calculus.

□

The cut rule for DLC^{μ} is the following:

$$\frac{\Delta \Rightarrow \Gamma \quad \Gamma \Rightarrow \Theta}{\Delta \Rightarrow \Theta} \quad \frac{\Delta([\,]) \Rightarrow \Gamma([\,]) \quad \Gamma([\,]) \Rightarrow \Theta([\,])}{\Delta([\,]) \Rightarrow \Theta([\,])}$$

Theorem 8. *The cut rule for the multisuccedent calculus is admissible.*

Proof. Let $\Delta, \Gamma, \Theta \in \mathcal{O}$. By i.h, $\Delta \Rightarrow \Gamma^{\bullet}$ and $\Gamma \Rightarrow \Theta^{\bullet}$. From the last hypersequent, we get $\Gamma^{\bullet} \Rightarrow \Theta^{\bullet}$. By application of cut, we obtain $\Delta \Rightarrow \Theta^{\bullet}$ which is equivalent by the previous lemma to $\Delta \Rightarrow_{\mu} \Theta$. Consider (cut_1). Again, by i.h, for every $C \in \mathcal{F}$ $\Delta(C) \Rightarrow (\Gamma(C))^{\bullet}$ and $\Gamma(C) \Rightarrow (\Theta(C))^{\bullet}$. From the last hypersequent, we get $(\Gamma(C))^{\bullet} \Rightarrow (\Theta(C))^{\bullet}$. By application of cut, we conclude that for every $C \in \mathcal{F}$ $\Delta(C) \Rightarrow (\Theta(C))^{\bullet}$ which is equivalent by the previous lemma to $\Delta([\,]) \Rightarrow_{\mu} \Theta([\,])$. □

3.2.3 Completeness for the full fragment

In order to get completeness for the full fragment (products and units), we use the concept of preordered 1-graded monoid.

Definition 21. *A preordered 1-graded monoid $\langle \widetilde{V}, I, J, \cdot, \hat{\,}\, ; \leq \rangle$ is a 1-graded monoid in which $\leq$ is a preorder*[5]*, and such that:*

- ***(sort preserving)*** *If $x \leq y$, x and y have the same sort.*

- ***(compatibility of the operations)*** *$\leq$ is compatible with $\cdot$ and $\hat{\,}$, i.e,*

$$\frac{a \leq a' \quad b \leq b'}{a \cdot b \leq a' \cdot b'} \quad \frac{a \leq a' \quad b \leq b'}{a \,\hat{}\, b \leq a' \,\hat{}\, b'}$$

Definition 22 (discontinuous preordered power-set frame)**.** *A discontinuous preordered power-set frame over a (preordered) 1-graded monoid $\langle \mathcal{P}(V_0 \cup (V_0 \cdot \$ \cdot V_0)), \circ_{\leq}, \backslash\backslash$*

[5]A preorder is a reflexive and transitive relation. If one adds symmetry, we get a partial order.

$, //, \ \hat{\circ}_{\leq}\ , \Uparrow, \Downarrow, \mathbb{I}, \mathbb{J};\ \subseteq\rangle = \mathbb{F}_{discont}\ .$

In order to define the new type interpretation, we need the concept of downward-closed subsets:

Definition 23 (downward-closed sets)**.** *Given a preordered set $(X, \leq)$, we say that $A \subseteq X$ is downward-closed (d.c), if:*

$$\forall x(\exists a(a \in A \,\&\, x \leq a) \rightarrow x \in A).$$

We define now the interpretation of types in discontinuous preordered power-set frames as:

Definition 24 (type interpretation in discontinuous preordered power-set frames)**.**

$$
\begin{aligned}
&[\![A]\!] \subseteq V_0,\ \textit{if } A\ \in \mathcal{A}_0,\ \textit{and such that } [\![A]\!] \textit{ is (d.c)} \\
&[\![A]\!] \subseteq V_1,\ \textit{if } A\ \in \mathcal{A}_1,\ \textit{and such that } [\![A]\!] \textit{ is (d.c)} \\
&[\![A \bullet B]\!] = [\![A]\!] \circ_{\leq} [\![B]\!] = \{e : \exists a, b\ a \in [\![A]\!] \,\&\, b \in [\![B]\!] \,\&\, e \leq a \cdot b\} \\
&[\![B/A]\!] = [\![B]\!]//[\![A]\!] \\
&[\![A\backslash B]\!] = [\![A]\!]\backslash\backslash[\![A]\!] \\
&[\![A \odot B]\!] = [\![A]\!]\ \hat{\circ}_{\leq}\ [\![B]\!] = \{e : \exists a_1 \cdot \$ \cdot a_2 \in [\![A]\!], b \in [\![B]\!] \wedge e \leq (a_1 \cdot \$ \cdot a_2) \,\hat{\cdot}\, b\} \\
&[\![B \uparrow A]\!] = [\![B]\!] \Uparrow [\![A]\!] \\
&[\![A \downarrow B]\!] = [\![A]\!] \Downarrow [\![B]\!] \\
&[\![I]\!] = \mathbb{I}_{\leq} = \{\delta : \delta \leq I\} \\
&[\![J]\!] = \mathbb{J}_{\leq} = \{\delta_1 \cdot \$ \cdot \delta_2 : \delta_1 \cdot \$ \cdot \delta_2 \leq \$\}
\end{aligned}
$$

The definition 24 is such that that all interpreted types are d.c:

Lemma 9. *For every $A \in \mathcal{F}$, $[\![A]\!]$ is d.c.*

Proof. By induction on the complexity of types. Atomic types and units are d.c by definition. $[\![A \bullet B]\!]$ and $[\![A \odot B]\!]$ are d.c, for $\circ_{\leq}$ and $\hat{\circ}_{\leq}$ are by definition d.c. Now, if $\delta' \leq \delta$ such that $\delta \in [\![A]\!]\backslash\backslash[\![B]\!]$, then $\delta' \in [\![A]\!]\backslash\backslash[\![B]\!]$. For, for every $a \in [\![A]\!]$,

$a \cdot \delta' \leq a \cdot \delta$ [6]. By definition of $\backslash\backslash$, $a \cdot \delta \in [\![B]\!]$, and by induction hypothesis (i.h), $[\![B]\!]$ is d.c. Thus, $a \cdot \delta' \in [\![B]\!]$ for every A, which proves that $\delta' \in [\![A]\!]\backslash\backslash[\![B]\!]$.
If $\delta_1' \cdot \$ \cdot \delta_2' \leq \delta_1 \cdot \$ \cdot \delta_2$ such that $\delta_1 \cdot \$ \cdot \delta_2 \in [\![B]\!] \Uparrow [\![A]\!]$ then $\delta_1' \cdot \$ \cdot \delta_2' \in [\![B]\!] \Uparrow [\![A]\!]$. For every $a \in [\![A]\!]$, by the compatiblity of $\hat{\cdot}$ and $\leq$, $(\delta_1' \cdot \$ \cdot \delta_2') \,\hat{\cdot}\, a \leq (\delta_1 \cdot \$ \cdot \delta_2) \,\hat{\cdot}\, a \in [\![B]\!]$. Thus, by i.h, $\delta_1' \cdot \$ \cdot \delta_2' \in [\![B]\!] \Uparrow [\![A]\!]$. Cases involving $//$ and $\Downarrow$ are completely similar. □

Definition 25 (discontinuous model over a preordered 1-graded monoid)**.** *A discontinuous preordered power-set model is a discontinuous power-set frame* $(\langle \mathcal{P}(V_0 \cup (V_0 \cdot \$ \cdot V_0)), \circ_{\leq}, \backslash\backslash, //, \hat{\circ}_{\leq}, \Uparrow, \Downarrow, \mathbb{I}_{\leq}, \mathbb{J}_{\leq}; \subseteq\rangle, [\![\cdot]\!]) = (\mathbb{F}_{discont}, [\![\cdot]\!])$ *with a valuation on the set of (continuous and discontinuous) types defined recursively over atomic variables of sort 0 or 1 as in definition 24.*

Theorem 9 (soundness of 1-DLC w.r.t the class of discontinuous models over preordered 1-graded monoids)**.**

Proof. Cases involving binary connectives are identical to the standard semantics (1-graded monoids). It remains to see the units rules. Right rules don't have any problems. Left rules hold, for $\leq$ is compatible with $\cdot$ and $\hat{\cdot}$. □

Our next step is to get completeness for the full calculus (products and units). Consider the construction of the canonical implicative model of theorem 7. We extend now the canonical model to the continuous and discontinuous product-types and units extending the universe (of the model) to configurations including all the possible types. The preorder of the canonical model is $\Rightarrow_\mu$*. Let's denote the canonical model* $\mathbb{CM}$*. Its valuation defined over atomic types is as follows:*

$$\forall A \in \mathcal{A},\ [\![A]\!] = \{\Delta : \Delta \in \& \vdash \Delta \Rightarrow_\mu A\}$$

Theorem 10 (preordered 1-graded monoid completeness)**.**

Theorem 11 (truth lemma)**.** *For every* $A \in \mathcal{F}$*:*

[6] By the compatibility $\cdot$ and $\leq$.

a) $\forall A \in \mathcal{A},\ [\![A]\!] = \{\Delta : \Delta \in \& \vdash \Delta \Rightarrow_\mu A\}$

b) $\forall \Delta \in \mathcal{O},\ \Delta \in [\![\Delta]\!]$

Remark 18. *If $A \in \mathcal{F}$ and $\Delta \in \mathcal{O}$, then by lemma 7, $\Delta \Rightarrow \overrightarrow{A}$ iff $\Delta \Rightarrow_\mu \overrightarrow{A}$.*

Proof. of theorem 11. The problem for products, say continuous product, is $\supseteq$. Let us suppose Δ such that $\Delta \Rightarrow A \bullet B$. But here, the proof doesn't fail. For, take $\Delta_A := A$ and $\Delta_B := B$. Then, there exists Δ_A and Δ_B, such that $\Delta \Rightarrow A \bullet B$. So, $\supseteq$ holds. Discontinuous product is completely similar.
Finally, consider units. By definition, $[\![I]\!] = \{\Delta : \Delta \Rightarrow_\mu I\}$, and $[\![J]\!] = \{\Delta_0, [\,], \Delta_1 : \Delta_0, [\,], \Delta_1 \Rightarrow_\mu \sqrt[0]{J}, [\,], \sqrt[1]{J}\}$.

□

As in theorem 7 we get completeness:

Theorem 12. *1-DLC and 1-DLCϵ are complete w.r.t the class of preordered discontinuous models.*

□

3.3 Conclusions and future work

We have seen several completeness results. This gives us coherence to the model-theoretic driven approach to the development of a calculus dealing with discontinuity. The incompleteness (see the problem with $J \stackrel{def}{=} I \uparrow I$.) w.r.t. the class of free 1-graded monoids leads us to conjecture *that completeness holds, if we replace J by $I \uparrow I$ in hypersequent rules involving the discontinuous unit J.*

*On the one hand, types inferences in the categorical calculus only involve types, whereas in 1-**DLC** (or 1-**DLC**ϵ), hypersequents (not in the case of the multisuccedent calculus) have an asymmetry between antecedents and succedents. We could have used the translation τ between the 1-discontinuous categorical calculus and the hypersequent calculus to explore completeness via the construction of the Lindenbaum algebra.*

Chapter 4

Linguistic applications and generative capacity

*In this chapter we present the 1-**DLC**ϵ at work. The definitions of signs, sort 0,1 lexicons, language recognition and string language recognition are given. Then, we present several linguistic phenomena in which discontinuity is involved. Particle verbs, discontinuous functors, medial extraction, cross-serial dependencies (in Dutch), gapping and quantifier raising are studied. Finally, the weak generative capacity of 1-**DLC**ϵ is related to Head Grammars [Pollard 1984] and the so-called mildly-context sensitive formalisms [Seki, et al 1991].*

4.1 Languages generated by 1-discontinuous Lambek grammars

Idea of the language generated by the 1-discontinuous Lambek calculus.

Definition 26 (lexicon of signs)**.** *A lexicon $\mathcal{L}ex$ is a finite set of signs $\{w_i - \mu_i : A_i\}_{i=1,\cdots,n}$, where w_i are respectively prosodic forms, μ_i semantic forms (λ-terms), and A_i a syntactic type. Moreover, the semantic type of μ_i is given by the homomorphism between syntactic types and semantic types, and the prosodic sort of w_i by the sort homomorphism.*

Example 3. *A lexicon:*

- $everyone - \mathbf{everyone} : (S \uparrow N) \downarrow S$
- $loves - \mathbf{love} : (N \backslash S)/N$
- $someone - \mathbf{someone} : (S \uparrow N) \downarrow S$
- $neither \cdot \$ \cdot nor - \lambda x \lambda y \lambda z \neg[(x\, z) \vee (y\, z)] : ((N\backslash S)/(N\backslash S)) \uparrow (N\backslash S)$

The semantic form **everyone** *has the following definition: $\lambda P \forall z[(\mathbf{person}\, z) \rightarrow (P\, z)]$ (similar λ-term for* **someone***). The last sign contains a prosodic form of sort 1 (and thus a syntactic type of sort 1).*

Definition 27 (sort 0 and 1 lexicon)**.** *A sort 0 lexicon $\mathcal{L}ex$ is a lexicon in which all syntactic types are of sort 0. A sort 1 $\mathcal{L}ex$ is a lexicon in which at least a sign has a syntactic type of sort 1.*

Definition 28 (language recognition)**.** *Given a lexicon $\mathcal{L}ex$ and a target symbol S (S is a type of arbitrary sort), we define the language generated by $\mathcal{L}ex$ and target symbol S $\mathcal{L}(\mathcal{L}ex, S)$ as $\{w - \mu : S|$ there exists an assignment of types for w, such that there exists a configuration Δ in which types occurring in Δ belong to the lexical assignment, and $\Delta \Rightarrow S\}$. μ is the semantics of one of the possible derivations of $\Delta \Rightarrow S$. The set of prosodic forms belonging to $\mathcal{L}(\mathcal{L}ex, S)$ is called the string language generated by $\mathcal{L}ex$.*

4.2 Linguistic applications: 1-DLCϵ at work

*We show 1-discontinuous lexicons and derivations in 1-**DLC**-ϵ. Examples and type assignments follow ideas from Morrill [1992, 1994, 2000], Solias [1992], and Morrill and Merenciano [1996].*

- ***Medial extraction**:*

‘man that Peter saw **e** yesterday’ *(1)*

$that - \mathbf{rel} : (CN\backslash CN)/((S\uparrow N)\odot I)$

$man - \mathbf{man} : CN$

$Peter - \mathbf{peter} : N$

$saw - \mathbf{see} : (N\backslash S)/N$

$yesterday - \mathbf{yesterday} : S\backslash S$

*The that-relative with medial extraction cannot be derived in **LC**[1]:*

*The derivation for (1) 1-**DLC**ϵ is:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{N,(N\backslash S)/N,N,S\backslash S\Rightarrow S}}{N,(N\backslash S)/N,[\,],S\backslash S\Rightarrow S\uparrow N}\uparrow R \qquad \dfrac{}{\Rightarrow I}\,IR}{N,(N\backslash S)/N,S\backslash S\Rightarrow (S\uparrow N)\odot I}\odot R \qquad CN,CN\backslash CN\Rightarrow CN}{CN,(CN\backslash CN)/((S\uparrow N)\odot I),N,(N\backslash S)/N,S\backslash S\Rightarrow S}\,/L$$

Remark 19. *The assignment* $that - \mathbf{rel} : (CN\backslash CN)/\hat{}(S\uparrow N)$ *works for the medial extraction as well:*

*The derivation for (1) 1-**DLC**$\hat{},\check{}$ is:*

[1] Only extraction at the periphery.

$$\dfrac{\dfrac{\dfrac{\vdots}{N,(N\backslash S)/N,N,S\backslash S\Rightarrow S}}{\dfrac{N,(N\backslash S)/N,[\,],S\backslash S\Rightarrow S\uparrow N}{N,(N\backslash S)/N,S\backslash S\Rightarrow\hat{}(S\uparrow N)}\,\hat{}R}\uparrow R \qquad CN,CN\backslash CN\Rightarrow CN}{CN,(CN\backslash CN)/\hat{}(S\uparrow N),N,(N\backslash S)/N,S\backslash S\Rightarrow S}\,/L$$

- ***Topicalisation***:

'Bill John knows Mary loves' *(2)*

Consider the (sort 0) lexicon:

$Bill$ – **bill** : $S/((S\uparrow N)\odot I)$

$John$ – **john** : N

$Mary$ – **mary** : N

$knows$ – **know** : $(N\backslash S)/S$

$loves$ – **love** : $(N\backslash S)/N$

$$\dfrac{S\Rightarrow S \qquad \dfrac{\dfrac{\vdots}{N,(N\backslash S)/S,N,(N\backslash S)/N,[\,]\Rightarrow S\uparrow N} \qquad \dfrac{}{\Rightarrow I}}{N,(N\backslash S)/S,N,(N\backslash S)/N\Rightarrow(S\uparrow N)\odot I}}{S/((S\uparrow N)\odot I),N,(N\backslash S)/S,N,(N\backslash S)/N,\Rightarrow S}$$

Remark 20. *Again, the unary operator* $\hat{}$ *can replace the use of the continuous unit* I*. So, for topicalisation we can type Bill with* $S/\hat{}(S\uparrow N)$*:*

$$\dfrac{S\Rightarrow S \qquad \dfrac{\dfrac{\vdots}{N,(N\backslash S)/S,N,(N\backslash S)/N,[\,]\Rightarrow S\uparrow N}}{N,(N\backslash S)/S,N,(N\backslash S)/N\Rightarrow\hat{}(S\uparrow N)}\,\hat{}R}{S/\hat{}(S\uparrow N),N,(N\backslash S)/S,N,(N\backslash S)/N,\Rightarrow S}$$

- ***discontinuous functors*** *Consider discontinuous functors as the following:*

'Mary rang John up' *(3)*

Consider the (sort 1) lexicon:

$rang \cdot \$ \cdot up - \textbf{phone} : (N\backslash S) \uparrow N$

$Mary - \textbf{mary} : N$

$John - \textbf{john} : N$

*The 1-**DLC** for (2) is:*

$$\dfrac{N \Rightarrow N \quad \overline{N, N\backslash S \Rightarrow S}}{N, \sqrt[0]{(N\backslash S) \uparrow N}, N, \sqrt[1]{(N\backslash S) \uparrow N} \Rightarrow S} \uparrow L$$

- ***Parenthetical adverbials***:

Consider the parenthetical adverbial example:

'La Maria sortosament agafa el tren' *(4)*

'La Maria agafa sortosament el tren' *(4')*

'La Maria agafa el tren sortosament' *(4")*

Consider the (sort 0) lexicon:

$sortosament - \textbf{always} : (S \uparrow I) \downarrow S$

$agafa - \textbf{take} : (N\backslash S)/N$

$La \cdot Maria - \textbf{maria} : N$

$el - \textbf{iota} : N/CN$

$tren - \textbf{train} : CN$

The derivation for (3) is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{N, (N\backslash S)/N, CN/N, N \Rightarrow S}}{N, I, (N\backslash S)/N, CN/N, N \Rightarrow S} I\, L}{N, [\,], (N\backslash S)/N, CN/N, N \Rightarrow S \uparrow I} \quad \overline{S \Rightarrow S}}{N, (S \uparrow I) \downarrow S, (N\backslash S)/N, CN/N, N \Rightarrow S} \downarrow L$$

Instead of the type assignment $(S \uparrow I) \downarrow S$ we could have used $(\breve{\ } S) \downarrow S$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{N, (N\backslash S)/N, CN/N, N \Rightarrow S}}{N, I, (N\backslash S)/N, CN/N, N \Rightarrow S}\,\breve{}R}{N, [\,], (N\backslash S)/N, CN/N, N \Rightarrow \breve{}S} \qquad \overline{S \Rightarrow S}}{N, (\breve{}S) \downarrow S, (N\backslash S)/N, CN/N, N \Rightarrow S}\downarrow L$$

Cases (3') and (3") are generated in the same manner. In these cases both type assignments (with unit and the split unary operator) work as well.

Remark 21. *The unary operator $\breve{}$ can avoid the use of the continuous unit I*

- ***Quantifier raising***:

The quantifier (we saw in chapter 1) $(S \uparrow N) \downarrow S$ works in subject position as well as in object position for everyone, everything. *Different readings (narrow, wide scope) are derived (in different derivations!).*

- ***Gapping constructions***:

'John Coltrane played the tenor saxophone and Elvin Jones, the drums' *(5)*

A possible type for and *is $(((S \uparrow TV) \odot TV)\backslash S)/(S \uparrow TV)$. Solias proposes another type assignment in terms of another type constructor that we are able to translate in 1-**DLC**ϵ with type $(((N \bullet J \bullet N) \odot TV)\backslash S)/(N \bullet J \bullet N)$. Observe the use of the discontinuous unit J.*

- ***Dutch subordinate clause cross-serial dependencies***

'Jan het boek wil kunnen lezen' *(5)*

('John wants to be able to read the book'*)*

Consider the following sort 1 lexicon:

$Jan - \mathbf{jan} : N$

$het - \mathbf{iota} : N/CN$

$boek - \mathbf{book} : CN$

$wil - \mathbf{want} : (N\backslash S_-) \downarrow (N\backslash S)$

$\$ \cdot kunnen - \mathbf{be\_able} : (N\backslash S_-) \downarrow (N\backslash S_-) =: K$

$\$ \cdot lezen - \mathbf{read} : N\backslash(N\backslash S_-) =: L$

*Here's a 1-**DLC** derivation of (5):*

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{N \Rightarrow N} \quad \overline{CN \Rightarrow CN}}{N/CN, CN \Rightarrow N}\ \backslash L \quad \dfrac{\overline{N \Rightarrow N} \quad \overline{\sqrt[0]{S_-}, [\,], \sqrt[1]{S_-} \Rightarrow \overrightarrow{S_-}}}{N, \sqrt[0]{N\backslash S_-}, [\,], \sqrt[1]{N\backslash S_-} \Rightarrow S_-}\ \backslash L}{N/CN, CN, \sqrt[0]{L}, [\,], \sqrt[1]{L} \Rightarrow \overrightarrow{N\backslash S_-}}\ \backslash L \quad \overline{\overrightarrow{N\backslash S_-} \Rightarrow \overrightarrow{N\backslash S_-}}}{N/CN, CN, \sqrt[0]{L}, \sqrt[0]{K}, [\,], \sqrt[1]{K}, \sqrt[1]{L} \Rightarrow \overrightarrow{N\backslash S_-}}\ \downarrow L \quad \dfrac{\overline{N \Rightarrow N} \quad \overline{S \Rightarrow S}}{N, N\backslash S \Rightarrow S}}{N, N/CN, CN, \sqrt[0]{L}, \sqrt[0]{K}, (N\backslash S_-) \downarrow (N\backslash S), \sqrt[1]{K}, \sqrt[1]{L} \Rightarrow S}\ \downarrow L$$

The derivational semantics (with the lexical semantics of the types) gives the expected logical form $((want\, be\_able(read\,(iota\,\mathbf{book})))\, j)$.

Remark 22. *In general we have omitted the derivational semantics via the Curry-Howard homomorphism (and with the lexical semantics). The reader is invited to check that the logical forms obtained are correct.*

4.3 On the weak generative capacity of 1-DLC grammars

*In this section we see that lexicalized Head Grammars (HG) (in an adapted version of Roach [1987]), are weakly-equivalent to 1-**DLC** grammars. Moreover, a fragment of 1-**DLC** grammars is shown to be weakly-equivalent to HG grammars. The class of head Grammars (invented by Pollard [84]) is a proper subclass of the class of Multiple Context-Free Grammars (MCFG) (Seki et al [1991]). The grammars of MCFG are context-free rewriting systems generated by linear (or regular) functions defined on sets of tuples of strings. These formal systems are mildly context sensitive formalisms (MCS formalisms), which are supposed to be a very promising (in cognitive terms) approach to natural language (see e.g. Stabler [2004]). MCSFs have the following properties:*

- *Universal polynomial recognition.*
- *Account of (limited) crossed-dependencies.*
- *The constant growth property, which means that if a MCS language L is ordered in an indexed union of sets $L = \bigcup L_i$ according to the length of the recognized strings, the length$(L_{i+1} - L_i)$ is bounded by a (finite) constant. So, for example a language like $\{a^{2^n}, n > 0\}$, which is recognized by unification grammars (say, an LFG grammar), cannot be a MCS language.*

A linear function f from $V^{k_1} \times \cdots \times V^{k_n} \to V^{j_1} \times \cdots \times V^{j_m}$, $f(\vec{x_1}, \cdots, \vec{x_n}) = (\vec{x_1}, \cdots, \vec{x_m})$, is such that every component of a tuple in the domain of f appears in the image by f at most once.

Example 4.

- $((x_1, x_2), (y_1, y_2)) \mapsto (x_1 y_1, y_2 x_2)$
- $(x_1, x_2) \mapsto (x_1 x_2)$
- $(x_1, x_2) \mapsto (a x_1 b, c x_2 d)$, *where a, b, c, d are string parameters.*

Definition 29. *A HG grammar is a context-free rewriting system with three operations* $\mathbf{conc_1}, \mathbf{conc_2}, \mathbf{wrap}$ *(functions) defined on nonterminal symbols NT and terminal symbols T. NT and T are represented by pairs.*

Let's see $\mathbf{conc_1}, \mathbf{conc_2}, \mathbf{wrap}$*:*

$$\mathbf{conc_1}((x_1, x_2), (y_1, y_2)) = (x_1, x_2 y_1 y_2)$$

$$\mathbf{conc_2}((x_1, x_2), (y_1, y_2)) = (x_1 x_2 y_1, y_2)$$

$$\mathbf{wrap}((x_1, x_2), (y_1, y_2)) = (x_1 y_1, y_2 x_2)$$

Example 5. *Let G be the HG grammar with the production rules:*

- $S \to conc_2((a, \epsilon), A) | \Lambda$

- $A \to wrap(S, (b, c))$

Implicitly, the non-terminals are assumed to be pairs.

Theorem 13. *Let G be the HG grammar with target symbol s with lexicalized production rules: $\{X_i \to wrap(\alpha_i, \beta_i)\}_{i=1,\ldots,n,\,n>0} \cup \{Z_j \to w_j\}_{j=1,\ldots,m,\,m>0}$, where at least α_i or β_i is a terminal. Then, the string language recognized by $\mathcal{L}(G)$ is generated by a 1-sort lexicon (1-**DLC**) grammar.*

Proof. For every unary rule $A \to w$, categorize the sort[2] 1 string w as A, i.e., $w : A$. In the case of binary rules ($\mathbf{conc_1}, \mathbf{conc_2}$, *and* $\mathbf{wrap}$,), say $X \to wrap(a, Y)$, categorize a as $a : X \uparrow Y$. The other case is symmetrical. Observe, that our sort 1 lexicon contains first-order types, so right rules of $\uparrow, \downarrow$ don't apply. It's easy to see that derivations of both systems are in a bijective correspondence. So, the string language generated by the 1-**DLC** lexicon is the same that $\mathcal{L}(G)$.

□

Theorem 14. *Let $\mathcal{L}ex$ be a sort 1 1-**DLC** lexicon, restricted to the connectives $\uparrow, \downarrow, \odot$ with sort functionalities $(1, 1) \to 1$. Then, $\mathcal{L}(\mathcal{L}ex, s)$ is recognised by a head grammar.*

Proof. The fragment 1-$\mathbf{DLC}_{\uparrow_{(1,1)\to 1}, \downarrow_{(1,1)\to 1}, \odot_{(1,1)\to 1}}$ is completely isomorphic to the continuous Lambek calculus LC. Map $\odot$ to $\bullet$, $\uparrow$ to $/$ and $\downarrow$ to $\backslash$. The implicit rule of associativity is in both calculi. Sort 1 atomic variables are mapped to a set (equipotent) of (sort 0) atomic variables. By the Pentus theorem [1993], there exists a context-free grammar G which recognizes the string language generated by a LC grammar. Translate the context-free grammar to the restricted case, and interpret concatenation by wrap. Thus, we are done. □

4.4 Conclusions

*1-**DLC**ϵ or (1-**DLC** $\hat{}, \check{}$) account for several discontinuity phenomena giving them the right semantic (or logical) forms. We have seen some results on the weak generative capacity of 1-**DLC** grammars, and there arises a suggestive idea: 1-**DLC** meets two of the three characteristic properties of MCS formalisms, namely the constant*

[2] Tuples of strings w are in natural bijection with prosodic forms of sort 1: $(x, y) \mapsto x \cdot \$ \cdot y$.

growth property and an adequate description of (limited) cross-serial dependencies. However, the universal polynomial recognition is not possible for the 1-$\boldsymbol{DLC}\epsilon$, for Pentus [2003] has proved that the Lambek calculus is NP-complete. We conjecture that the full 1-$\boldsymbol{DLC}\epsilon$ has still the weak generative capacity of Head Grammars. Following Pentus' ideas on interpolation and the multiplicative property of 1-$\boldsymbol{DLC}\epsilon$[3], may be a good point of departure. Finally, a remark on the so-called strong generative capacity (SGC). SGC is the set of structural descriptions of a language. Usually, structural descriptions are interpreted in terms of trees. Here however, we posit that cut-free proof derivations modulo permutations of the application of rules (not affecting the derivational semantics) are the real structural descriptions of 1-$\boldsymbol{DLC}\epsilon$.

[3] Basic mathematical tools to prove that Lambek grammars are context-free.

Chapter 5

Conclusions

*A pure logical calculus without structural rules has been presented: the 1-**DLC**ϵ. We think according to the data and the mathematical results, that the 1-discontinuous Lambek calculus is adequate to account for a range of discontinuity phenomena. Moreover, the calculus seems to meet the criteria of the so-called mildly context sensitive formalisms, namely Head Gramars. Moreover, the extension of the Lambek calculus to hypersequent calculus preserves the elegancy of the former calculus, for the mathematical results of 1-DLC are in some sense parallel to the Lambek calculus (semantic results and proof-theoretical results).*

*Processing issues in LC, have shown that (cut-free) proof nets or the abstract representation of cut-free derivations are a very elegant method to study acceptability and incremental processing of sentences. Morrill [Morrill 2000] shows how a variety of linguistic phenomena including garden-pathing, left to right scope preference in quantified sentences and center embedding unacceptability are explained in terms of a simple complexity metric measuring the number of open dependencies at each step of processing of the sentence. In Bott, Valentin [2004], LC proofnets are used for the study of the incremental semantic interpretation of generic indefinites. In Morrill [Morrill 2003], where discontinuity is considered, the delay of principle B effect is explained in terms of proofnets for the ω-**DLC** (see chapter 1). It seems that 1-DLCϵ and maybe ω-DLCϵ will have a proofnet machinery similar to the LC case.*

The use of the continuous and discontinuous units has shown to be very useful:

- *The bridge unary operator could be simulated with the units:* $\hat{}S \stackrel{def}{=} S \odot I$, *for* S *of sort 1.*

- *The split unary operator could be simulated with the units:* $\check{}S \stackrel{def}{=} S \uparrow I$, *for* S *of sort 0.*

- *The connective "sequence" (see gapping constructions in chapter 4) could be simulated with the units:* $A \Diamond B \stackrel{def}{=} A \bullet J \bullet B$.

5.1 Future work

We could have considered (following [Morrill 2002]), a hybrid 1-DLCϵ with multiple different types of prosodic separators $\$_v$. *Types would be at most 1-sorted, but more fine-grained because new (sort 1) continuous and discontinuous connectives could be used (e.g.,* $\downarrow_v$ *for* $v = 1, \ldots, n$, $n \geq 0$*). We conjecture that the results obtained for the 1-DLCϵ can be extended in a natural way to the ω-DLCϵ (joint work with Morrill and Fadda). The reader will have noticed that the logic studied in this work is essentially associative. As a matter of fact, the LC is known as the* associative *Lambek Calculus. We claim that the 1-**DLC** with or without units could be called the* associative *1-discontinuous Lambek calculus. Several authors in the field (Moortgat [1995] and Morrill [1994]) have realized that an associative type-logical grammar may lead to problems of overgeneration. Non-associativity has been studied in depth in type-logical grammar, so it remains to explore the use of non-associativity with something similar to a hypersequent calculus.*

Type-logical grammar has shown that unary connectives (the so-called "brackets") are very useful to prevent the violation of some of the famous Ross constraints (See Morrill [1994]). "Brackets" are useful to control associativity. We think that these tools have (model-theoretically) their place in sorted 1-graded monoids, and hence in the hypersequent calculus.

*Finally, we could go beyond pairs of strings. It should be checked the viability of calculi similar to the 1-**DLC**, using other residuated connectives defined in terms of more complex linear functions of tuples of strings.*

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